

Quadratic Extension Algebras and Quaternion Algebras Over Fields (of Characteristic not 2)

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1 Prologue

Tsit Yuen Lam, *Introduction to Quadratic Forms over Fields*, the end of section III.5,
“Characterizations of Quaternion Algebras”:

The second characterization of quaternion algebras is in terms of the canonical involution “bar” on such algebras. Note that this is an involution “of the first kind”: that is, it is an involution that restricts to the identity map on the center F of the algebra.

Theorem 5.2. *Let $B \neq F$ be a finite-dimensional simple F -algebra with center F equipped with an F -algebra involution of the first kind $x \mapsto \bar{x}$ such that $x + \bar{x} \in F$ and $x\bar{x} \in F$ (for all $x \in B$). Then B is isomorphic to a quaternion algebra over F .*

Although this is a very nice characterization of quaternion algebras, it will not be needed in the rest of this book. Therefore, we’ll leave its proof as an exercise to the reader. The main idea of the proof is that, given the properties of the involution $x \mapsto \bar{x}$, every element $x \in B$ satisfies a quadratic equation over the center F ; namely

$$x^2 - x(x + \bar{x}) + x\bar{x} = 0 \quad (x + \bar{x}, x\bar{x} \in F).$$

From all properties of the F -algebra B mentioned in the quoted text we shall single out the property “every element $x \in B$ satisfies a quadratic equation over F ”, and shall try

to find out how far it gets us. The idea is to gather enough information about such algebras, so that by imposing on them some additional conditions, preferably as weak as possible, we will be able to characterize quaternion algebras.

We kicked this particular stone that rested peacefully at the top of the hill, and it started rolling and tumbling down the slope. We are going to run after it, because we are curious to see where it will come to rest.

Ready? Here we go.

2 Some conventions, terminology, and notation

All fields mentioned in this text are assumed to have the characteristic different from two. In particular, K will always denote a field with $\text{char } K \neq 2$. Unless specified otherwise, every algebra is over a field and is silently assumed to be associative, unital, and non-trivial (i.e., it does not consist of the zero alone, which also serves as the unity).

We will encounter some algebras that do not conform to the default assumptions. The **zero algebra** (over some field) is the set $\{0\}$ equipped with the evident operations; the solitary element 0 of the zero algebra is the neutral element both for addition and multiplication. A **zero-product algebra** is a vector space A equipped with the all-zero multiplication, that is, $ab = 0$ for all $a, b \in A$; a zero-product algebra is associative, and it is unital if and only if it is the zero algebra. A **zero-square algebra** is an associative algebra A in which $a^2 = 0$ for every $a \in A$; it is unital if and only if it is the zero algebra. Two other (equivalent) descriptions of a zero-square algebra: an anti-commutative associative algebra; an associative Lie algebra.

Let A be an associative algebra, not necessarily with unity, over a field K . We **adjoin an identity element (unity)** to A to obtain the algebra $K \oplus A$; writing $(\lambda, a) \in K \oplus A$ as a formal sum $\lambda \oplus a$, the multiplication on $K \oplus A$ is defined by

$$(\lambda \oplus a)(\mu \oplus b) := \lambda\mu \oplus (\lambda b + \mu a + ab) .$$

The algebra $K \oplus A$ is associative with the unity $1 \oplus 0$.¹ If the algebra A already possesses a unity 1_A , then $0 \oplus 1_A$ is an idempotent of the unital algebra $K \oplus A$ that is different from its unity.

¹Adjoining an identity element is a universal construction. Let K be a field, let \mathbf{Alg}_K be the category of associative unital K -algebras with morphisms the functions preserving addition, multiplication by scalars, and multiplication, and sending the unity to the unity; also, let \mathbf{Alg}'_K be the category of associative K -algebras whose morphisms are functions preserving addition, multiplication by scalars, and multiplication. Let $\mathbf{Alg}_K \rightarrow \mathbf{Alg}'_K$ be the functor which ‘forgets the unity’; the left adjoint $A \mapsto K \oplus A$ to this functor does the job of adjoining identity elements, and for each algebra A in \mathbf{Alg}'_K the mapping $A \rightarrow K \oplus A : a \mapsto 0 \oplus a$ is a universal arrow in \mathbf{Alg}'_K from A to the forgetful functor.

Whenever S is (the underlying set of) a structure containing a distinguished element called “zero” and denoted by 0 , and X is any subset of S , then we write $X^\bullet := X \setminus \{0\}$.

If S is a structure, part of which is a multiplicatively written associative binary operation with a neutral element (and there is only one such operation in the structure), which therefore endows S with the structure of a monoid, then we denote this monoid by S^\cdot (note the small dot) and by S^\times the multiplicative group of invertible elements of S^\cdot . In particular, if S is a ring, or an algebra, then S^\times denotes the group of units (multiplicatively invertible elements) of S .

If S is a structure, part of which is an additively written binary operation which makes S an abelian group (examples of such structures are rings, algebras, modules, \dots), then we denote this abelian group by S^+ .

Let S be a structure, part of which is a multiplicatively written binary operation. For every subset X of S we shall denote by $\square X$ the set of squares xx of all elements x of X . If S is a structure for which we have also defined X^\bullet for $X \subseteq S$, or S^\times , or both, then we agree that $\square X^\bullet$ means $\square(X^\bullet)$, and $\square S^\times$ means $\square(S^\times)$; that is, $(-)^\bullet$ and $(-)^\times$ have precedence over $\square(-)$.

3 Quadratic extension algebras

We define a **quadratic extension K -algebra**² as a non-zero associative K -algebra A with unity $1 = 1_A$, where every $x \in A$ satisfies a quadratic equation $x^2 + \alpha x + \beta = 0$ for some $\alpha, \beta \in K$. Let A be a quadratic extension K -algebra. As usual we identify $K1_A$ with K (we can do this because $1_A \neq 0$) so that $K \subseteq A$, and refer to the elements of K as **scalars**. We set $\vec{A} := \{0\} \cup \{u \in A \setminus K \mid u^2 \in K\}$, call \vec{A} the **pure part** of A , and say that elements of \vec{A} are the **pure elements** of A .

Lemma 1. *Every element x of a quadratic extension K -algebra A has a unique representation of the form $x = \alpha + u$ with $\alpha \in K$ and $u \in \vec{A}$.*

Proof. If $x = \alpha \in K$, then $x = \alpha + 0$ is a required representation, which is unique: if $x = \beta + u$ with $\beta \in K$ and $u \in \vec{A}$, then $u = \alpha - \beta \in K \cap \vec{A} = \{0\}$.

Suppose $x \notin K$. Then $x^2 + \beta x + \gamma = 0$ for some $\beta, \gamma \in K$, thus $(x + \frac{1}{2}\beta)^2 = \frac{1}{4}\beta^2 - \gamma$, which means that $x = \alpha + u$ with $\alpha = -\frac{1}{2}\beta \in K$ and $u^2 = \frac{1}{4}\beta^2 - \gamma \in K$; since $u \in K$ implies $x \in K$, we must have $u \notin K$ because $x \notin K$, and hence $u \in \vec{A}$. Uniqueness. Suppose that also $x = \alpha' + u'$ with $\alpha' \in K$ and $u' \in \vec{A}$. Then $u' = (\alpha - \alpha') + u$, where $(u')^2 = (\alpha - \alpha')^2 + u^2 + 2(\alpha - \alpha')u \in K$ and $u^2 \in K$, thus $\delta := (\alpha - \alpha')u \in K$, so we must have $\alpha - \alpha' = 0$, since otherwise $u = \delta/(\alpha - \alpha') \in K$, contradicting $u \notin K$. \square

²Also known as a K -algebra of degree 2.

We will occasionally introduce (or represent) an element of a quadratic extension K -algebra as $\alpha + u$ (or $\beta + v$, or $\lambda + g$, ...) without any comment; in such a case we will silently assume that this is the unique *scalar + pure* representation of the element.

Lemma 1 says that the pure part \vec{A} of a quadratic extension K -algebra A is a set of representatives of the cosets of the subgroup K^+ of the additive group A^+ . But more is true — \vec{A} is a very special set of representatives.

Lemma 2. *The pure part of a quadratic extension K -algebra A is a K -subspace of A .*

Proof. If $\lambda \in K$ and $u \in \vec{A}$, then $\lambda u \in \vec{A}$. This is clear if $\lambda = 0$ or $u = 0$; if $\lambda \neq 0$ and $u \neq 0$, then $u \notin K$, hence also $\lambda u \notin K$, and $(\lambda u)^2 = \lambda^2 u^2 \in K$.

Given any $u, v \in \vec{A}$, we shall show that $u + v \in \vec{A}$.

If u and v are linearly dependent, then $u + v = \lambda u$ for some $\lambda \in K$ or $u + v = \mu v$ for some $\mu \in K$ (or both), and in either case $u + v \in \vec{A}$ by what we have proved above.

Now let u and v be linearly independent.

We claim that $1, u$, and v are linearly independent; in particular, $u + v \notin K$. Suppose, to the contrary, that $\alpha + \beta u + \gamma v = 0$ for some $\alpha, \beta, \gamma \in K$, not all zero. Then $\alpha \neq 0$ because u and v are linearly independent, hence at least one of β, γ is non-zero, say $\gamma \neq 0$, and we have $v = -\alpha/\gamma - (\beta/\gamma)u$, which is a *scalar + pure* representation of v ; since the unique such representation of v is $0 + v$, we must have $\alpha = 0$, a contradiction.

There exist $\alpha_1, \alpha_2 \in K$ such that $\beta_1 := (u + v - \alpha_1)^2 \in K$ and $\beta_2 := (u - v - \alpha_2)^2 \in K$, whence we can express $uv + vu$ in two ways as a linear combination of $1, u$, and v :

$$\begin{aligned} uv + vu &= (\beta_1 - u^2 - v^2 - \alpha_1^2) + 2\alpha_1 u + 2\alpha_1 v \\ &= (-\beta_2 + u^2 + v^2 + \alpha_2^2) - 2\alpha_2 u + 2\alpha_2 v. \end{aligned}$$

Comparing the coefficients at u and v we find that $\alpha_1 = -\alpha_2$ and $\alpha_1 = \alpha_2 = -\alpha_1$, thus $\alpha_1 = 0$ and hence $(u + v)^2 = \beta_1 \in K$; since $u + v \notin K$, we have $u + v \in \vec{A}$. \square

We see that every quadratic extension K -algebra A splits as $A = K \oplus \vec{A}$, where \vec{A} is a subspace (actually a hyperplane) of the K -space A , and $\square \vec{A} \subseteq K$. This suggests the following definition.

A **ground Clifford K -algebra** is an associative K -algebra of the form $A = K \oplus V$, where V is a subspace of the K -space A and $\square V \subseteq K$.

Recall that a **Clifford K -algebra** is a pair (A, V) , where A is a non-zero unital associative K -algebra, V is a K -subspace of A , $\square V \subseteq K1_A = K$, and V generates A (i.e., V is a set of generators of the K -algebra A). More often than not we refer to (A, V) simply as a Clifford K -algebra A , knowing that the subspace V is somewhere at hand.

If (A, V) is a Clifford K -algebra, and A properly contains K (hence $V \not\subseteq K$), then $K \cap V = 0$. To prove this, consider any $\alpha \in K \cap V$, and pick $v \in V \setminus K$; since the sum $\alpha + v$ belongs to V , its square $(\alpha + v)^2 = (\alpha^2 + v^2) + 2\alpha v$ is a scalar, thus αv is a scalar, which is possible only if $\alpha = 0$. When $A = K$, the subspace V is either 0 or K ; in this special case we always choose $V = 0$ instead of $V = K$.

A Clifford K -algebra A thus always contains the K -subspace $K \oplus V$, which is a foundation on which A is built as the set of all finite sums of scalars and of elements of V and of products of two, three, four, \dots elements of V ; when V has a finite dimension n , we can stop with products of n elements of V . A ground Clifford algebra, as defined above, is a Clifford algebra that consists of nothing else but its foundation; when we are building such an algebra from its foundation, we never get off the ground.

Let $A = K \oplus V$ be a ground Clifford K -algebra. We denote by \square_V the quadratic functional $V \rightarrow K : v \mapsto v^2$. Having \square_V , we erect the free Clifford K -algebra $Cl(\square_V)$, which contains $K \oplus V$ as a K -subspace. Elements of V have the same squares in the algebra $Cl(\square_V)$ as in the algebra A . However, when $\dim V > 1$, the algebra $Cl(\square_V)$ properly contains the algebra A , and the multiplication of elements of V in $Cl(\square_V)$ differs from their multiplication in A . There exists a unique K -algebra homomorphism $h : Cl(\square_V) \rightarrow A$ which fixes all elements of A ; the homomorphism h collapses the free Clifford algebra $Cl(\square_V)$ onto the ground Clifford algebra A .

Lemma 3. *A quadratic extension K -algebra A is a ground Clifford K -algebra $K \oplus \vec{A}$. A ground Clifford algebra $A = K \oplus V$ is a quadratic extension algebra with $\vec{A} = V$.*

Proof. The first statement is just a restatement of Lemma 2.

For the second statement, suppose $A = K \oplus V$ is a ground Clifford algebra. Let $x = \alpha + v$, where $\alpha \in K$ and $v \in V$, be an arbitrary element of A . Clearly x satisfies the quadratic equation $x^2 - 2\alpha x + (\alpha^2 - v^2) = 0$. Since $x^2 = (\alpha^2 + v^2) + 2\alpha v$ is a scalar if and only if $\alpha v = 0$, that is, if and only if $\alpha = 0$ or $v = 0$, it follows that $\vec{A} = V$. \square

Therefore, if A is a ground Clifford algebra, it is a quadratic extension algebra, the split $A = K \oplus V$ such that $\square_V \subseteq K$ is unique, the subspace $V = \vec{A}$ is determined by A , so we can write \square_V as \square_A without ambiguity. Just keep in mind that the quadratic functional \square_A is defined only on the pure part of A , not on the whole A .

How about subalgebras of quadratic extensions algebras? The following is an immediate consequence of the definitions of a quadratic extension algebra and of its pure part.

Lemma 4. *Let A be a quadratic extension K -algebra. Every K -subalgebra B of A is a quadratic extension K -algebra, where $\vec{B} = B \cap \vec{A}$.*

4 The conjugation and the (reduced) norm

Let A be a quadratic extension K -algebra.

Every $x \in A$ has a unique representation $x = \alpha + u$ with $\alpha \in K$ and $u \in \vec{A}$; we shall call $\tau_A x = \tau x := \alpha$ the **scalar part** of x and $\pi_A x = \pi x := u$ the **pure part** of x . The mappings $\tau, \pi: A \rightarrow A$ are complementary K -linear projectors onto the subspaces K resp. \vec{A} of the K -space A : $\tau A = K$, $\pi A = \vec{A}$, $\tau^2 = \tau$, $\pi^2 = \pi$, $\tau\pi = \pi\tau = 0$, and $\tau + \pi = \text{id}_A$.

For every $\alpha + u \in A$ we set $(\alpha + u)^* := \alpha - u$, and call the mapping $A \rightarrow A: x \mapsto x^*$ the **conjugation** on A . The conjugation is the endomorphism $\tau - \pi$ of the K -space A ; it is evidently an involution, that is, $x^{**} = x$ for every $x \in A$, so it is in fact an automorphism of the K -space A . For every $x \in A$ we have $2\tau x = x + x^*$ and $2\pi x = x - x^*$, therefore $x \in K$ iff $x^* = x$ and $x \in \vec{A}$ iff $x^* = -x$.

Lemma 5. *If $u, v \in \vec{A}$, then $(uv)^* = vu$.*

Proof. Let $uv = \alpha + w$, where $\alpha \in K$ and $w \in \vec{A}$. Since $uv + vu = (u + v)^2 - u^2 - v^2$ is a scalar, we have $vu = \beta - w$, where $\beta = uv + vu - \alpha \in K$. We will show that $\beta = \alpha$.

Suppose that $w \neq 0$. The product $uv \cdot vu = u \cdot v^2 \cdot u = u^2 v^2$ is a scalar; because $uv \cdot vu = (\alpha + w)(\beta - w) = (\alpha\beta - w^2) + (\beta - \alpha)w$, we must have $(\beta - \alpha)w = 0$, which implies $\beta - \alpha = 0$.

Now suppose that $w = 0$, thus $uv = \alpha$ and $vu = \beta$. If $u = 0$, then $uv = vu = 0$, so suppose that $u \neq 0$. We have $\alpha u = u\alpha = u \cdot uv = u^2 v$ and $\beta u = vu \cdot u = vu^2 = u^2 v$, thus $\alpha u = \beta u$, whence $\alpha = \beta$. \square

Lemma 6. *For all $x, y \in A$ we have $(xy)^* = y^* x^*$.*

Proof. Splitting $x = \alpha + u$ and $y = \beta + v$, we have $(xy)^* = (\alpha\beta + \alpha v + \beta u + uv)^* = (\alpha\beta)^* + (\alpha v)^* + (\beta u)^* + (uv)^* = \alpha\beta - \alpha v - \beta u + vu = y^* x^*$. \square

If $x, y \in A$ and $\alpha := x + y \in K$, then x and $y = \alpha - x$ commute; in particular, x and x^* always commute. For every $x \in A$ we call $N_A(x) = N(x) := xx^* = x^* x \in K$ the **(reduced) norm of x** . If $x = \alpha + u$, then $N(x) = \alpha^2 - u^2$. For a pure u we have $N(u) = -u^2$, that is, denoting the restriction of N_A to \vec{A} by ν_A , we have $\nu_A = -\square_A$.

Lemma 7. *The norm $N_A: A \rightarrow K$ is a homomorphism of multiplicative monoids.*

Proof. Clearly $N(1) = 1$. If $x, y \in A$, then $N(xy) = (xy)(xy)^* = xy y^* x^* = x \cdot y y^* \cdot x^* = x x^* \cdot y y^* = N(x) N(y)$. \square

Lemma 8. *An element x of A is invertible if and only if $N(x) \neq 0$.*

Proof. If x is invertible, then $N(x)N(x^{-1}) = N(1) = 1$, hence $N(x) \neq 0$. Conversely, if $N(x) \neq 0$, then x is invertible with the inverse $x^*/N(x)$. \square

The following proposition summarizes some properties of conjugation.

Proposition 9. *The conjugation $x \mapsto x^*$ of a quadratic extension algebra A is an involutive anti-automorphism of A which fixes every scalar; moreover, $x + x^*$ and xx^* are scalars for every $x \in A$.*

We can characterize quadratic extension algebras by existence of a ‘conjugation’. First, a preparatory lemma.

Lemma 10. *Let $x, x' \in A$, and suppose that x is not a scalar, while $x + x'$ and xx' are scalars. Then $x' = x^*$.*

Proof. Splitting $x = \alpha + u$, $x' = \beta - u$, we have a scalar $xx' = (\alpha\beta - u^2) + (\beta - \alpha)u$, thus $\beta - \alpha = 0$ because $u \neq 0$. \square

Note that if $x, x' \in A$ are such that $x + x'$ is a scalar, and x is a scalar, then x' is a scalar. Also, given a scalar x , both $x + x'$ and xx' are scalars for any scalar x' .

And now, here is the promised characterization.

Proposition 11. *Let A be a non-zero unital associative K -algebra, and let $A \rightarrow A : x \mapsto \bar{x}$ be a function such that $x + \bar{x}$ and $x\bar{x}$ are scalars for every $x \in A$. Then A is a quadratic extension K -algebra, and $\bar{x} = x^*$ for every $x \in A \setminus K$. Consequently, if $\bar{\alpha} = \alpha$ for every $\alpha \in K$, then $\bar{x} = x^*$ for every $x \in A$.*

Proof. Every $x \in A$ satisfies the equation $x^2 - x(x + \bar{x}) + x\bar{x} = 0$, which proves that A is a quadratic extension K -algebra. If $x \in A \setminus K$, then $\bar{x} = x^*$ by Lemma 10. \square

5 Examples of quadratic extension algebras

Every quaternion algebra $(\frac{a,b}{K})$, where $a, b \in K^\bullet$, is a quadratic extension K -algebra.

The field K itself is a quadratic extension K -algebra whose pure part is 0.

The quadratic extension K -algebras A with a one-dimensional $\vec{A} = Ku$ are precisely the free Clifford algebras generated by one-dimensional quadratic spaces. The kind of algebra we get depends on the nature of the scalar u^2 . If $u^2 = 0$, then A is the algebra of ‘dual scalars’ over the field K . If u^2 is a square of a non-zero scalar, then A is isomorphic to the K -algebra $K \times K$. Finally, if $u^2 = \alpha$ is not a square of a scalar, then A is (isomorphic to) the quadratic extension field $K(\sqrt{\alpha})$ of K .

Next on our agenda are the quadratic extension K -algebras A with a two-dimensional pure part; we shall determine all of them (that is, all their isomorphism classes).

Let (e, f) be an orthogonal basis of the quadratic space (\vec{A}, \square_A) . The symmetric bilinear functional associated with the quadratic functional $\square_A: \vec{A} \rightarrow K: u \mapsto u^2$ is

$$B_A(u, v) := \frac{1}{2}((u+v)^2 - u^2 - v^2) = \frac{1}{2}(uv + vu) = \frac{1}{2}(uv + (uv)^*) = \tau(uv).$$

Since $ef + fe = 2B_A(e, f) = 0$, we see that $\tau(ef) = 0$, which means that ef is pure. We have $e^2 = \alpha$, $f^2 = \beta$, $ef = -fe = \gamma e + \delta f$ for some scalars $\alpha, \beta, \gamma, \delta$. The necessary and sufficient conditions $(ee)f = e(ef)$, $(ef)e = e(fe)$, \dots for associativity of A give us the equations $\alpha = \delta^2$, $\beta = \gamma^2$, and $\alpha\gamma = \beta\delta = \gamma\delta = 0$.

When $\alpha \neq 0$, we have $\delta \neq 0$ and $\beta = \gamma = 0$ (the case $\beta \neq 0 \implies \gamma \neq 0 \implies \alpha = \delta = 0$ is analogous). Renaming e/δ as e , we obtain a basis (e, f) of \vec{A} such that $e^2 = 1$, $f^2 = 0$, and $ef = -fe = f$. Let the matrices $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ have entries in K ; they form a basis of the underlying K -space of the K -algebra $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ of the upper triangular 2×2 matrices with entries in K . The matrices E and F satisfy the equations $E^2 = I$, $F^2 = 0$, and $EF = -FE = F$, thus the K -linear mapping $A \rightarrow R$ which sends $1, e, f$ respectively to I, E, F is an isomorphism of K -algebras. Every matrix $X = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \in R$ has the scalar part $\tau X = \frac{1}{2}(\alpha + \gamma) = \frac{1}{2} \operatorname{tr} X$ (actually $\tau X = (\frac{1}{2} \operatorname{tr} X) \cdot I$), the norm $N(X) = \alpha\gamma = \det X$, and the conjugate $X^* = \begin{bmatrix} \gamma & -\beta \\ 0 & \alpha \end{bmatrix} = \tilde{X}$ (where for any positive integer n and any matrix $M \in \mathbb{M}_n(K)$, \tilde{M} denotes the adjugate of the matrix M , that is, the transpose of the matrix of cofactors of M). The pure part of R is $\vec{R} = \{X \in R \mid \operatorname{tr} X = 0\}$, that is, \vec{R} consists of all matrices of the form $\begin{bmatrix} \alpha & \beta \\ 0 & -\alpha \end{bmatrix} = \alpha E + \beta F$ with $\alpha, \beta \in K$. The quadratic functional \square_A is isomorphic to the quadratic functional $p_2: K^2 \rightarrow K: (x, y) \mapsto x^2$, and the free Clifford algebra $Cl(p_2)$ — isomorphic to the K -algebra $(\frac{1,0}{K})$ — has a basis $(1, i, j, k)$ with the multiplication table $i^2 = 1$, $ij = -ji = k$, $ik = -ki = j$, $j^2 = k^2 = jk = kj = 0$. The ‘collapsing’ homomorphism of K -algebras $h: Cl(p_2) \rightarrow A$, which sends i to e and j to f , sends $k = ij$ to $ef = f$, thus the kernel of h is the one-dimensional subspace $K \cdot (k - j)$ of $Cl(p_2)$.

The only other case is $\alpha = \beta = 0$, when also $\delta = \gamma = 0$; now \vec{A} is a zero-product algebra, and adjoining 1 to it we obtain the algebra A .

We have found that there are only two isomorphism classes of three-dimensional quadratic extension K -algebras: one class is of the K -algebra obtained by adjoining the unity to the zero-product algebra on K^2 , and the other class is of the K -algebra of the upper triangular 2×2 matrices with entries in K .

Digressing for a couple of moments, let us consider an arbitrary three-dimensional K -algebra A . We have just discussed the case where every element of A satisfies a quadratic equation over K . Otherwise, there is an element a of A that does not satisfy any quadratic equation. Since A is three-dimensional, a does satisfy some cubic equation, thus $f(a) = 0$ for some monic polynomial $f(X) \in K[X]$ of degree 3. Moreover, every polynomial $g(X) \in K[X]$ such that $g(a) = 0$ is divisible by $f(X)$. The elements $1, a, a^2$ of A are linearly independent, and $A = K \oplus Ka \oplus Ka^2 \cong K[X]/f(X)K[X]$. We know how the structure of A can be worked out in detail, depending on factorization of $f(X)$ into irreducible factors; we won't go into this, we just point out that A is commutative. So we have this nice little result: a three-dimensional K -algebra that is not commutative is isomorphic to the K -algebra of upper-triangular 2×2 K -matrices. Since the latter K -algebra possesses divisors of zero, we have also the following corollary: if a three-dimensional K -algebra is a domain, then it is a field.³

As the last example we provide a generous supply of rather degenerate quadratic extension algebras: if we adjoin 1 to any zero-square K -algebra B , we obtain a quadratic extension K -algebra $A = K \oplus B$ whose pure part is B ; in particular, we may choose B to be any zero-product K -algebra.

There do exist zero-square algebras that are not zero-product algebras. For example, the K -algebra $B := Ke \oplus Kf \oplus Kg$ with the multiplication table $e^2 = f^2 = g^2 = 0$ and $ef = -fe = fg = -gf = ge = -eg = e + f + g$ is a zero-square algebra. Choosing the basis (e, f, ef) of B instead of the basis (e, f, g) , we discover that the quadratic extension K -algebra $A := K \oplus B$ is isomorphic to the exterior algebra $\Lambda(K^2)$; put slightly differently, A is isomorphic to the free Clifford algebra $Cl(o_2)$ of the zero quadratic functional o_2 on K^2 , where $Cl(o_2)$ can be also presented as $\left(\begin{smallmatrix} 0,0 \\ K \end{smallmatrix}\right)$.

The variety of zero-square K -algebras is closed under taking direct products, subalgebras, and homomorphic images. The not-completely-trivial zero-square K -algebra $\left(\begin{smallmatrix} 0,0 \\ K \end{smallmatrix}\right)$ generates a subvariety of this variety which contains a great many not-completely-trivial zero-square K -algebras.⁴

³Every finite-dimensional algebra is algebraic, and every algebraic algebra that is a domain (has no divisors of zero) is a division ring.

⁴Question: does $\left(\begin{smallmatrix} 0,0 \\ K \end{smallmatrix}\right)$ generate the entire variety of zero-square K -algebras?

6 Pure calculus on a quadratic extension algebra

When I first came across Hamilton’s quaternions, they appeared to me as a not entirely natural marriage of the real line and the oriented three-dimensional Euclidean space. Every quaternion was a sum of a scalar and a vector, and the product of two such sums $s + x$ and $t + y$ was given by the formula

$$(s + x)(t + y) = (st - \langle x, y \rangle) + (tx + sy + x \times y) \quad (1)$$

which was a concoction of several ingredients of the algebra of three-dimensional vectors: a product of two scalars, two products of a vector by a scalar, a scalar/dot product of vectors, a vector/cross products of vectors, a sum of two scalars, and a sum of three vectors. For quite a while I thought that this was how the quaternions had actually been designed. I was mistaken, of course. In the beginning there were quaternions, quadruples of real numbers, created by Hamilton on October 16 of the year 1843, as a generalization of complex numbers (after almost 13 years — some say that there were ‘only’ 8 years — of futile attempts to set up such a generalization with triples of real numbers). When the quaternions were 40 years old, give or take a year or two, they were mercilessly hacked into their scalar and vector parts, and then these chunks (still dripping blood) were used to build the vector calculus of the oriented three-dimensional Euclidean space:

It is probably true that Hamilton spent too much time on quaternions. He did little else until his death in 1865, and few mathematicians shared his enthusiasm. Nevertheless quaternions changed the course of mathematics, though not in the way Hamilton intended. In the 1880s Josiah Willard Gibbs and Oliver Heaviside created what we now know as vector analysis, essentially by separating the real (“scalar”) part of a quaternion from its imaginary (“vector”) part. Hamilton’s followers were outraged to see the simple and elegant quaternions torn limb from limb, but the idea caught on with physicists and engineers, and it still holds sway today.

[John Stillwell, *Mathematics and Its History*, Second Edition, pp. 402–403.]

We are now going to follow in steps of Gibbs and Heaviside, constructing an analogue of (the algebraic part of) the “vector calculus” from the scalar and pure parts of elements of an arbitrary quadratic extension algebra; this will be the “pure calculus” of this section’s title. To get the right idea where and how to start, we take a look at the product of pure Hamilton’s quaternions (that is, vectors) x and y , which is

$$xy = -\langle x, y \rangle + x \times y ,$$

where the first summand is the scalar part, and the second summand is the vector part, of the product.

Let A be a quadratic extension K -algebra.

For any two pure elements u and v of A we define

$$\begin{aligned}\langle u, v \rangle &:= -\tau(uv) = -\frac{1}{2}(uv + vu) = -B_A(u, v) \in K, \\ u \times v &:= \pi(uv) = \frac{1}{2}(uv - vu) \in \vec{A}.\end{aligned}$$

We should perhaps write the two operations as $\langle u, v \rangle_A$ and $u \times_A v$ to avoid confusion — but we will always know what we are doing, won't we? Using the **scalar product** $\langle -, - \rangle$ and the **pure product** $-\times-$ of pure elements, the product of any two elements $\alpha + u$ and $\beta + v$ of A can be written in the *scalar + pure* form as

$$(\alpha + u)(\beta + v) = (\alpha\beta - \langle u, v \rangle) + (\beta u + \alpha v + u \times v), \quad (2)$$

which is (of course) precisely the no-longer-so-strange-looking formula (1). Note that the scalar product is a bilinear functional on \vec{A} , and that the pure multiplication is a bilinear operation on \vec{A} ; moreover, $\langle -, - \rangle$ is symmetric, $\langle v, u \rangle = \langle u, v \rangle$, while $-\times-$ is anticommutative, $v \times u = -u \times v$. Note that

$$\langle u, u \rangle = -u^2 = N(u)$$

for every pure u , thus $\langle -, - \rangle$ is the symmetric bilinear functional associated with the quadratic functional $\nu_A = -\square_A$ on the subspace \vec{A} (which is the restriction of the quadratic functional N_A defined on the space A).

Let $u, v, w \in \vec{A}$.

We say that u and v are **orthogonal**, and write $u \perp v$, if $\langle u, v \rangle = 0$. Since $\langle u, v \rangle = -\tau(uv) = -\tau(vu)$, we have the following equivalences:

$$u \perp v \iff uv \text{ is pure} \iff vu \text{ is pure} \iff uv = u \times v.$$

Since $\langle u, v \rangle = -\frac{1}{2}(uv + vu)$, u and v are orthogonal if and only if they anticommute, that is, if and only if $vu = -uv$. Similarly, since $u \times v = \frac{1}{2}(uv - vu)$, we have $u \times v = 0$ if and only if u and v commute.

Clearly $u \times u = 0$. Slightly more generally, if u and v are linearly dependent, then $u = \lambda v$ for some scalar λ or $v = \mu u$ for some scalar μ (or both), and in either case $u \times v = 0$. Equivalently, if $u \times v \neq 0$, then u and v are linearly independent.

The pure product is half the additive commutator in an associative algebra, thus (\vec{A}, \times) is a Lie algebra, hence $-\times-$ satisfies the Jacobi's identities:

$$\begin{aligned}u \times (v \times w) + v \times (w \times u) + w \times (u \times v) &= 0, \\ (u \times v) \times w + (v \times w) \times u + (w \times u) \times v &= 0.\end{aligned}$$

Here is another easily obtainable identity: on the one hand $N(uv) = N(u)N(v) = (-u^2)(-v^2) = u^2v^2$, and on the other hand $N(uv) = (\tau(uv))^2 - (\pi(uv))^2$, therefore

$$u^2v^2 = \langle u, v \rangle^2 - (u \times v)^2 . \quad (3)$$

For a few moments assume that V is some vector space over K , that $\langle -, - \rangle$ is a K -bilinear functional on V (not necessarily symmetric), and that $- \times -$ is a K -bilinear operation on V (not necessarily anticommutative). Define the bilinear multiplication $(x, y) \mapsto xy$ on $K \oplus V$ by⁵

$$(\alpha \oplus u)(\beta \oplus v) := (\alpha\beta - \langle u, v \rangle) \oplus (\beta u + \alpha v + u \times v) . \quad (4)$$

Let us compute the associator of $x_i = \alpha_i \oplus u_i \in K \oplus V$, $i = 1, 2, 3$:

$$\begin{aligned} (x_1 x_2) x_3 - x_1 (x_2 x_3) &= (-\langle u_1 \times u_2, u_3 \rangle + \langle u_1, u_2 \times u_3 \rangle) \\ &\quad \oplus ((u_1 \times u_2) \times u_3 - \langle u_1, u_2 \rangle u_3 - u_1 \times (u_2 \times u_3) + \langle u_2, u_3 \rangle u_1) . \end{aligned}$$

We see that the multiplication defined on $K \oplus V$ is associative if and only if

$$\begin{aligned} \langle u_1 \times u_2, u_3 \rangle &= \langle u_1, u_2 \times u_3 \rangle , \\ (u_1 \times u_2) \times u_3 - u_1 \times (u_2 \times u_3) &= \langle u_1, u_2 \rangle u_3 - \langle u_2, u_3 \rangle u_1 \end{aligned}$$

for all $u_1, u_2, u_3 \in V$. Assuming that (V, \times) is a Lie algebra, we can rewrite the left hand side of the second identity above as

$$\begin{aligned} (u_1 \times u_2) \times u_3 - u_1 \times (u_2 \times u_3) &= (u_1 \times u_2) \times u_3 + (u_2 \times u_3) \times u_1 \\ &= -(u_3 \times u_1) \times u_2 . \end{aligned}$$

In our case the multiplication on $A = K \oplus \vec{A}$ is associative and (\vec{A}, \times) is a Lie algebra, so the following identities hold for all $u, v, w \in \vec{A}$:

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle , \quad (5)$$

$$(u \times v) \times w = \langle u, w \rangle v - \langle v, w \rangle u , \quad (6)$$

$$u \times (v \times w) = \langle u, w \rangle v - \langle u, v \rangle w ; \quad (7)$$

⁵This is in fact the most general situation. Meaning what? Let $(x, y) \mapsto xy$ be any bilinear operation on $K \oplus V$ with the neutral element (multiplicative identity) $1 \oplus 0$. Denoting by $\kappa: K \oplus V \rightarrow K$ and $\varphi: K \oplus V \rightarrow V$ the projections onto the direct summands, we set $\langle u, v \rangle := -\kappa(uv)$ and $u \times v := \varphi(uv)$ for any $u, v \in V$; then the product $(\alpha \oplus u)(\beta \oplus v)$ is computed according to formula (4).

identity (7) follows from identity (6) because $u \times (v \times w) = -(-w \times v) \times u = (w \times v) \times u$ and because $\langle -, - \rangle$ is symmetric. Using the identities (5) and (6), we can derive the identity

$$\langle u_1 \times u_2, v_1 \times v_2 \rangle = \begin{vmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle \end{vmatrix}, \quad (8)$$

which holds for all $u_1, u_2, v_1, v_2 \in \vec{A}$:

$$\begin{aligned} \langle u_1 \times u_2, v_1 \times v_2 \rangle &= \langle (u_1 \times u_2) \times v_1, v_2 \rangle \\ &= \langle \langle u_1, v_1 \rangle u_2 - \langle u_2, v_1 \rangle u_1, v_2 \rangle \\ &= \langle u_1, v_1 \rangle \langle u_2, v_2 \rangle - \langle u_2, v_1 \rangle \langle u_1, v_2 \rangle. \end{aligned}$$

Taking $u_1 := v_1 := u$ and $u_2 := v_2 := v$ in the identity (8), we obtain the identity (3).

We define the **mixed product**, a trilinear functional $\langle -, -, - \rangle$ on \vec{A} , by

$$\langle u, v, w \rangle := \langle u \times v, w \rangle = \langle u, v \times w \rangle,$$

for all $u, v, w \in \vec{A}$. Since clearly $\langle u, u, v \rangle = \langle u, v, v \rangle = 0$ for all $u, v \in \vec{A}$, the mixed product is an *alternating* trilinear functional, thus also $\langle u, v, u \rangle = 0$ for all $u, v \in \vec{A}$, and

$$\langle u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3} \rangle = \text{sgn}(\sigma) \cdot \langle u_1, u_2, u_3 \rangle$$

for all $u_1, u_2, u_3 \in \vec{A}$ and every permutation σ of $\{1, 2, 3\}$. If $u, v, w \in \vec{A}$ are linearly dependent, then $\langle u, v, w \rangle = 0$; indeed, if, say, $w = \alpha u + \beta v$ for some $\alpha, \beta \in K$, then

$$\langle u, v, w \rangle = \langle u, v, \alpha u + \beta v \rangle = \alpha \cdot \langle u, v, u \rangle + \beta \cdot \langle u, v, v \rangle = 0.$$

Equivalently, if $\langle u, v, w \rangle \neq 0$, then u, v, w are linearly independent.

For all $u, v \in \vec{A}$ we have $\langle u, u \times v \rangle = \langle u, u, v \rangle = 0$ and also $\langle v, u \times v \rangle = 0$, that is,

$$u \perp u \times v, \quad v \perp u \times v.$$

We conclude the section with the identity

$$\langle u \times v, v \times w, w \times u \rangle = \langle u, v, w \rangle^2,$$

which holds for all pure u, v , and w :

$$\begin{aligned} \langle u \times v, v \times w, w \times u \rangle &= \langle (u \times v) \times (v \times w), w \times u \rangle \\ &= \langle \langle u, v \times w \rangle v - \langle v, v \times w \rangle u, w \times u \rangle \\ &= \langle \langle u, v, w \rangle v, w \times u \rangle \\ &= \langle u, v, w \rangle \cdot \langle v, w \times u \rangle \\ &= \langle u, v, w \rangle \cdot \langle v, w, u \rangle \\ &= \langle u, v, w \rangle^2. \end{aligned}$$

7 The ‘grand identity’

Let A be a quadratic extension K -algebra.

Our aim in this section is to prove the identity

$$\begin{vmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \langle u_1, v_3 \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \langle u_2, v_3 \rangle \\ \langle u_3, v_1 \rangle & \langle u_3, v_2 \rangle & \langle u_3, v_3 \rangle \end{vmatrix} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle, \quad (9)$$

which holds for all $u_i, v_i \in \vec{A}$, $i = 1, 2, 3$.⁶

This time we cannot prove the identity by a calculation straightaway. We need some preparations, in the course of which we will happen upon quaternion algebras.

Lemma 12. *Let $u, v_1, v_2, v_3 \in \vec{A}$. If $\langle u, v_i \rangle \neq 0$ for some $i = 1, 2, 3$, then*

$$\langle v_1, v_2, v_3 \rangle \cdot u = \langle u, v_1 \rangle (v_2 \times v_3) + \langle u, v_2 \rangle (v_3 \times v_1) + \langle u, v_3 \rangle (v_1 \times v_2). \quad (10)$$

Proof. Write $d := \langle v_1, v_2, v_3 \rangle$, and denote by w the pure element represented by the right hand side of (10). Easy calculations show that $\langle w, v_i \rangle = \langle u, v_i \rangle \cdot d$ for $i = 1, 2, 3$, and that $u \times w = 0$. For every $v \in \vec{A}$ we have $0 = (u \times w) \times v = \langle u, v \rangle w - \langle w, v \rangle u$, that is, $\langle w, v \rangle u = \langle u, v \rangle w$. Taking $v = v_1, v_2, v_3$, we get $\langle u, v_i \rangle \cdot du = \langle u, v_i \rangle \cdot w$ for $i = 1, 2, 3$, and it follows that $du = w$ because by assumption at least one $\langle u, v_i \rangle$ is non-zero. \square

Lemma 13. *Suppose that $v_1, v_2, v_3 \in \vec{A}$ are such that $\langle v_1, v_2, v_3 \rangle \neq 0$. Then (v_1, v_2, v_3) is a basis of \vec{A} . For every non-zero $u \in \vec{A}$ at least one of $\langle u, v_1 \rangle, \langle u, v_2 \rangle, \langle u, v_3 \rangle$ is $\neq 0$.*

Proof. We know that v_1, v_2, v_3 are linearly independent because $d := \langle v_1, v_2, v_3 \rangle \neq 0$. Similarly $w_1 := v_2 \times v_3, w_2 := v_3 \times v_1, w_3 := v_1 \times v_2$ are linearly independent because $\langle w_1, w_2, w_3 \rangle = d^2$. Clearly $\langle v_i, w_j \rangle = \delta_{ij}d$ for $i, j = 1, 2, 3$. Let V be the subspace of \vec{A} spanned by the v_i ’s, and let W be the subspace of \vec{A} spanned by the w_i ’s. Consider an arbitrary $u \in \vec{A}$. If $\langle u, v_1 \rangle \neq 0$, then $u \in W$ by Lemma 12. If $\langle u, v_1 \rangle = 0$, then $\langle u + w_1, v_1 \rangle = d \neq 0$, thus $u + w_1$ lies in W , and so does u . It follows that $\vec{A} = W$. Since $V \subseteq \vec{A} = W$ and V is three-dimensional, we must have $V = W = \vec{A}$.

Regarding the second statement of the lemma, consider any non-zero $u \in \vec{A}$. Since $\vec{A} = W$, we have $u = \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3$ for some scalars ξ_1, ξ_2, ξ_3 , which are not all zero because u is non-zero. But $\langle u, v_i \rangle = \xi_i \langle w_i, v_i \rangle = \xi_i d$ for $i = 1, 2, 3$, so at least one $\langle u, v_i \rangle$ is non-zero. \square

⁶Suppose that $\vec{A} = K^3$, that $\langle u, v \rangle$ is the usual scalar product and $u \times v$ is the usual vector product of triples $u, v \in K^3$, so that $\langle u, v, w \rangle$ is the determinant of the matrix with the rows (or columns) u, v, w . Denote by U the matrix with the rows u_1, u_2, u_3 and by V the matrix with the columns v_1, v_2, v_3 ; then the identity (9) says that $\det(UV) = \det(U) \det(V)$.

An equivalent formulation of the second statement of Lemma 13: if $u, v_1, v_2, v_3 \in \vec{A}$ and $u \neq 0$ and $\langle u, v_i \rangle = 0$ for each $i = 1, 2, 3$, then $\langle v_1, v_2, v_3 \rangle = 0$.

Lemma 14. *The identity (10) holds for all $u, v_1, v_2, v_3 \in \vec{A}$.*

Proof. If $u = 0$, the identity is $0 = 0$. If $u \neq 0$ and $\langle u, v_i \rangle = 0$ for each $i = 1, 2, 3$, then $\langle v_1, v_2, v_3 \rangle = 0$ and once more the identity is $0 = 0$. If $\langle u, v_i \rangle \neq 0$ for some $i = 1, 2, 3$, then the identity holds by Lemma 12. \square

Proof of identity (9). Write $w_1 := v_2 \times v_3$, $w_2 := v_3 \times v_1$, $w_3 := v_1 \times v_2$. We expand the determinant D on the left hand side of (9) along the first row, applying (8) to the cofactors of the row's entries as we go, then after some rearranging we apply (10) with u_1 in place of u , and we are almost there:

$$\begin{aligned}
D &= \langle u_1, v_1 \rangle \cdot \langle u_2 \times u_3, w_1 \rangle + \langle u_1, v_2 \rangle \cdot \langle u_2 \times u_3, w_2 \rangle + \langle u_1, v_3 \rangle \cdot \langle u_2 \times u_3, w_3 \rangle \\
&= \langle u_2 \times u_3, \langle u_1, v_1 \rangle w_1 + \langle u_1, v_2 \rangle w_2 + \langle u_1, v_3 \rangle w_3 \rangle \\
&= \langle u_2 \times u_3, \langle v_1, v_2, v_3 \rangle u_1 \rangle \\
&= \langle v_1, v_2, v_3 \rangle \cdot \langle u_2 \times u_3, u_1 \rangle \\
&= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle .
\end{aligned}$$

\square

8 Quadratic spaces of arbitrary dimension

In the next three sections (after this one) we shall carry out a classification (of sorts) of quadratic extension algebras. At the highest level we shall classify quadratic extension algebras according to the rank of their norm. With every quadratic extension algebra A there are associated the quadratic space (A, N_A) and its quadratic subspace (\vec{A}, ν_A) . Denoting by ε the quadratic functional $\xi \mapsto \xi^2$ on K , we have $N_A = \varepsilon \perp \nu_A$, and hence $\text{rank } N_A = \text{rank } \nu_A + 1$. We shall say that the rank of N_A is the rank of A and write it $\text{rank } A$, and similarly, that the rank of ν_A is the rank of \vec{A} and write it $\text{rank } \vec{A}$. We shall find that a quadratic extension algebra can have only one of the three ranks 1, 2, or 4, and that the quadratic extension algebras of rank 4 are precisely the quaternion algebras.

Surely you have noticed that we are not restricting ourselves to just the finite-dimensional quadratic extension algebras: they may be of any dimension (if they can), finite or infinite, and the same is true of the associated quadratic spaces. Consequently, we will require, for our classification task, a few simple facts about quadratic spaces of arbitrary dimension.

Let V be a vector space over a field K , of finite or infinite dimension.

A functional $q: V \rightarrow K$ is said to be **quadratic** if there exists a bilinear functional b on V such that $q(x) = b(x, x)$ for every $x \in V$. Setting $B(x, y) := \frac{1}{2}(b(x, y) + b(y, x))$, we have $q(x) = B(x, x)$, where B is a symmetric bilinear functional; B is uniquely determined by q , namely $B(x, y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$.⁷ Let E be a basis of V . Every $x \in V$ has a unique representation $x = \sum_{e \in E} \xi_e e$ with only finitely many coordinates $\xi_e \in K$ different from 0. If $x = \sum_e \xi_e e$ and $y = \sum_e \eta_e e$, then $B(x, y) = \sum_{e, f} B(e, f) \xi_e \eta_f$, where $B(e, f) = B(f, e)$ for all $e, f \in E$. Save for the symmetry, the coefficients $B(e, f)$ are completely arbitrary: if β_{ef} ($e, f \in E$) are any scalars such that $\beta_{ef} = \beta_{fe}$ for all basis vectors e and f , then, with x and y as above, the formula $B(x, y) := \sum_{e, f} \beta_{ef} \xi_e \eta_f$ defines a symmetric bilinear functional on V .

Given a quadratic functional q and the associated symmetric bilinear functional B , both on V , we have a **quadratic space** (V, q, B) . We define quadratic subspaces, orthogonality $x \perp y$, orthogonal sums of subspaces $U \perp W$, orthogonal complements U^\perp , and so on, just as for the finite-dimensional quadratic spaces. The radical of the quadratic space V is also defined in the same way, namely $\text{rad } V := V^\perp$, and V is said to be regular if $\text{rad } V = 0$. We define the rank of the quadratic space V to be the codimension of its radical: $\text{rank } V := \text{codim } \text{rad } V = \dim(V/\text{rad } V)$. The most significant difference between finite-dimensional and infinite-dimensional quadratic spaces is this: for a regular V , the linear mapping $V \mapsto V^\wedge : x \mapsto B(x, -)$ is an isomorphism of vector spaces if V is finite-dimensional, while it is only injective if V is infinite-dimensional.⁸

We will need a few facts about quadratic space of an arbitrary dimension.⁹

Proposition 15. *Let (V, q, B) be a quadratic space over K . If U is a maximal regular subspace of V , then $V = U \perp U^\perp$ and $U^\perp = V^\perp$. If U is a complementary subspace of the radical V^\perp in the space V , then U is a maximal regular subspace of V .*

Proof. Let U be a maximal regular subspace of V . First, $U \cap U^\perp = \text{rad } U = 0$ because U is regular. In order to show that $U + U^\perp = V$, consider any $v \in V$. If $v \in U$, we are done, so assume that $v \notin U$. Since by maximality of U the quadratic subspace $U + Kv$ is not regular, its radical contains a non-zero vector u' , which lies in U^\perp so it certainly does not belong to U ; but then $U + Kv = U + Ku'$, and we have $v \in U + Ku' \subseteq U + U^\perp$. From $U \subseteq V$ we get $U^\perp \supseteq V^\perp$. As for the reverse inclusion, first notice that every

⁷Now I must have written down this formula for the N -th time, where N is a not-so-small natural number. It is no longer such an exhilarating experience as it was the first time round.

⁸If the dimension m of V is infinite, then the dimension of V^\wedge is $|K|^m \geq 2^m > m$, therefore V is not isomorphic to V^\wedge .

⁹We shall actually use only Proposition 17 for $\dim U = 1$ and Proposition 18 with $n = 2$.

non-zero $u' \in U^\perp$ is isotropic, since otherwise $U \perp Ku'$ would be regular, contrary to maximality of U ; but then U^\perp is totally isotropic, thus every $u' \in U^\perp$ is orthogonal to all vectors in U^\perp as well as to all vectors in U , and it follows that $U^\perp \subseteq V^\perp$.

Now let U be any complementary subspace of V^\perp in V ; it is clear that $V = U \perp V^\perp$. Let U' be the radical of the quadratic subspace U . The subspace U' of V is orthogonal to U as well as to V^\perp , thus is orthogonal to V , hence is contained in V^\perp , and it follows that $U' = 0$, which proves that U is regular. Suppose that a regular subspace U_1 of V contains U . Then $U_1 = U \perp (U_1 \cap V^\perp)$ (since the lattice of subspaces of V is modular), where $U_1 \cap V^\perp$ is contained in the radical of the quadratic subspace U_1 , and it follows that $U_1 \cap V^\perp = 0$ and hence $U_1 = U$; this proves maximality of U . \square

Corollary 16. *The dimension of every maximal regular subspace of a quadratic space is equal to the rank of the quadratic space.*

Proposition 15 not only proves that maximal regular subspaces of a quadratic space V exist, it also characterizes them as the complementary subspaces of the radical V^\perp . The existence part relies on Zorn's lemma (lurking in the background). Suppose we have a subspace U of a K -space V and want to prove that U has a complementary subspace in V . First we notice that the set \mathcal{W} of all subspaces W of V such that $U \cap W = 0$, ordered by inclusion, is strictly inductive (this is because the union of any nonempty chain in \mathcal{W} belongs to \mathcal{W}), so it has maximal elements; then it is easy to see that every maximal member of \mathcal{W} is a complementary subspace of U . Similarly we prove, using Zorn's lemma, that every regular subspace of a quadratic space V is contained in a maximal regular subspace. To this end we only need to prove that the union of a nonempty chain \mathcal{U} of regular subspaces of V is a regular subspace of V : every $u \in \text{rad} \bigcup \mathcal{U}$ lies in some $U \in \mathcal{U}$ and belongs to $\text{rad} U$, hence $u = 0$ since U is regular.

Proposition 17. *If U is a finite-dimensional regular subspace of a quadratic space (V, q, B) over K , then $V = U \perp U^\perp$.*

Proof. First we have $U \cap U^\perp = \text{rad} U = 0$. In order to prove that $U + U^\perp = V$, consider any $v \in V$. The mapping $U \rightarrow K : u \mapsto B(v, u)$ is a linear functional on U ; since U is finite-dimensional and regular, there exists $v' \in U$ so that $B(v, u) = B(v', u)$ for every $u \in U$. But then $v'' := v - v' \in U^\perp$, thus $v = v' + v'' \in U + U^\perp$. \square

Proposition 18. *If (V, q, B) is a quadratic space over K , then for every natural number $n \leq \text{rank } V$ there exists an n -dimensional regular subspace of V .*

Proof. The proof is by induction on n . When $n = 0$, the assertion is evidently true. Now suppose $n > 0$. By induction hypothesis there exists an $(n-1)$ -dimensional regular

subspace U of V . We have $V = U \perp U^\perp$ by Proposition 17. Since $\dim U < n \leq \text{rank } V$, the regular subspace U is not a maximal regular subspace of V , according to Corollary 16. It follows that U^\perp is not totally isotropic, since otherwise every subspace U_1 of V that properly contains U would have a radical containing $U_1 \cap U^\perp \neq 0$ (this because we have $U_1 = U \perp (U_1 \cap U^\perp)$). Thus we can choose an anisotropic $u' \in U^\perp$ and construct the n -dimensional regular subspace $U \perp Ku'$ of V . \square

9 Quadratic extension algebras of rank 1

In this section we examine the structure of a quadratic extension K -algebra A of rank 1.

Since $\text{rank } \nu_A = 0$, the quadratic space (\vec{A}, ν_A) is totally isotropic. This means that $\langle u, v \rangle = 0$ for all pure u and v , so the product of any two pure elements is pure, hence the subspace \vec{A} with the induced multiplication is a zero-square algebra (which, however, is not a subalgebra of A , because it does not contain the multiplicative identity 1_A of A). The algebra A is obtained from the zero-square algebra \vec{A} by adjoining the multiplicative identity to it. The zero-square algebra \vec{A} can be chosen arbitrarily: adjoining an identity element to a zero-square algebra always yields a quadratic extension algebra of rank 1.

The subspace \vec{A} is clearly an ideal of A , so we have the following result:

A quadratic extension K -algebra A of rank 1 is simple if and only if $A = K$.

This is the only result about quadratic extension algebras of rank 1 that we will need later on to establish various characterizations of quaternion algebras. However, we are curious about the structure of quadratic extension algebras of rank 1, or, which is the same thing, we want to know how to construct arbitrary zero-square algebras.

So let S be any zero-square K -algebra. Since $xy + yx = (x + y)^2 - x^2 - y^2 = 0$ for all $x, y \in S$, the multiplication on S is anticommutative, thus the product $xy = \frac{1}{2}(xy - yx)$ is at the same time a Lie bracket. The algebra S is nilpotent; to be precise, $SSS = 0$. Indeed, any $x, y, z \in S$ satisfy the Jacobi's identity

$$xyz + yzx + zxy = 0 ,$$

and since $y(zx) + (zx)y = 0$, it follows that $xyz = 0$.

Let us define the **annihilator** of S as $\text{ann } S := \{x \in S \mid xS = 0\}$; because of the anticommutativity of multiplication we have also $\text{ann } S = \{x \in S \mid Sx = 0\}$. It is clear that $\text{ann } S$ is a subspace of the K -space S . We can rewrite $SSS = 0$ as $SS \subseteq \text{ann } S$.

Set $W := \text{ann } S$, and let V be any complementary subspace of W in the K -space S . We have an antisymmetric bilinear function $V \times V \rightarrow W : (v, v') \mapsto vv'$, which completely

determines the multiplication on $S = V \oplus W$, because for all $v, v' \in V$ and all $w, w' \in W$ we have $(v+w)(v'+w') = vv'$. Moreover, $\{v \in V \mid vV = 0\} = 0$; in particular, V is not one-dimensional, because then $\{v \in V \mid vV = 0\} = V \neq 0$.

Conversely, suppose we have K -spaces V and W and an antisymmetric bilinear function $\mu: V \times V \rightarrow W$. We define a bilinear multiplication on $S := V \oplus W$ by

$$(v \oplus w)(v' \oplus w') := 0 \oplus \mu(v, v') .$$

The K -algebra S is associative because $(SS)S = S(SS) = 0$, and it is a zero-square algebra because $\mu(v, v) = 0$ for all $v \in V$. Moreover, $\text{ann } S = \{v \in V \mid \mu(v, V) = 0\} \oplus W$, thus if we have chosen μ so that $\{v \in V \mid \mu(v, V) = 0\} = 0$, then $\text{ann } S = 0 \oplus W$.

We will not go into detailed classification of zero-square algebras, because this would not be very interesting, and what is more, it would be of no use for our purpose.

A final note. Let $A \neq K$ be a quadratic extension algebra of rank 1. Then $\vec{A} \neq 0$ and also $W := \text{ann } \vec{A} \neq 0$; indeed, were $W = 0$, we would have $\vec{A}\vec{A} \subseteq W = 0$, therefore $W = \text{ann } \vec{A} = \vec{A} \neq 0$, a contradiction. Let us split \vec{A} as $V \oplus W$. If \vec{A} is not a zero-product algebra, then $\dim V \geq 2$. It is easy to see that it is possible to have $\dim V = 2$ and $\dim W = 1$, and that in such a case the K -algebra A is isomorphic to $\left(\frac{0,0}{K}\right)$.

10 Quadratic extension algebras of rank 2

In this section A is a quadratic extension K -algebra of rank 2.

Now $\text{rank } \nu_A = 1$, so there exists $u \in \vec{A}$ such that $\alpha := u^2$ is a non-zero scalar, and $\vec{A} = Ku \perp V$, where $V = u^\perp = \text{rad } \vec{A}$. Every $v \in V$ anticommutes with every $w \in \vec{A}$ because $v \perp w$. Since $(\xi u + v)^2 = \xi^2 u^2 + \xi(uv + vu) + v^2 = \alpha \xi^2$ for all $\xi \in K$ and all $v \in V$, we have $\square \vec{A} = \alpha \cdot \square K$, thus A determines the element $\alpha \cdot \square K^\times$ of the group $K^\times / \square K^\times$.

Let $v, v' \in V$. We claim that $vv' = 0$. First, $(vv')^2 = vv' \cdot vv' = -v'v^2v' = 0$. Since u anticommutes with v as well as with v' , u commutes with vv' . Since $v \perp v'$, the product vv' is pure, hence $vv' = \xi u + w$ for some $\xi \in K$ and some $w \in V$, and we have $0 = (vv')^2 = \alpha \xi^2$, thus $\xi = 0$ and $vv' = w$. Since u both commutes and anticommutes with w , we have $uw = wu = -uw$ and hence $uw = 0$, which implies that $w = 0$ because u is invertible, and hence $vv' = w = 0$, as claimed. The subspace V is certainly closed under multiplication, which is all-zero on V .

Let $v \in V$. We claim that $uv \in V$ (and hence $vu = -uv \in V$). This is an immediate consequence of $u \perp v$: the product uv is pure, and $u \perp uv$ because $uv = u \times v$.

We see that V is an ideal of A , thus A can be simple only if $V = 0$. Moreover, if $V = 0$, then $A = K + Ku$ is simple if and only if α is a non-square, and in such a case A is isomorphic to the quadratic extension field $K(\sqrt{\alpha})$ of the field K .

A quadratic extension K -algebra of rank 2 is simple if and only if it is a quadratic extension field of K .

As was the case with quadratic extension algebras of rank 1, the result above is all we will need to know about quadratic extension algebras of rank 2 for the purpose of characterizations of quaternion algebras, and as was the case with quadratic extension algebras of rank 1, we nevertheless want to know more about the structure of quadratic extension algebras of rank 2. This time we will pursue the classification all the way to the isomorphism classes; though there's no real need to go into so much detail, we will do it simply because we *can* do it, and quite painlessly at that.

The linear transformation $\varphi: V \rightarrow V: v \mapsto uv$ satisfies the identity $\varphi^2 = \alpha \cdot \text{id}_V$ because $u(uv) = u^2v = \alpha \cdot v$ for every $v \in V$.

Conversely, assume that we have a vector space A over K , two elements 1 and u of A and a subspace V of A such that $A = K1 \oplus Ku \oplus V$ (this is an internal direct sum), and that we have also a non-zero scalar α and a linear transformation $\varphi: V \rightarrow V$. Then there is one and only one bilinear multiplication $(x, y) \mapsto xy$ on A which has the following properties:

- ◇ $1x = x1 = x$ for every $x \in A$;
- ◇ $u^2 = \alpha$;
- ◇ $uv = \varphi v$ and $vu = -\varphi v$ for every $v \in V$;
- ◇ $vv' = 0$ for all $v, v' \in V$.

If, in addition, φ satisfies the identity $\varphi^2 = \alpha \cdot \text{id}_V$, then the vector space A , equipped with the multiplication, is a quadratic extension K -algebra of rank 2 with $\vec{A} = Ku \oplus V$. Since $(\xi u + v)^2 = \xi^2 \alpha \in K$ for all $\xi \in K$ and $v \in V$, it remains to prove associativity of multiplication. It suffices to verify that $(w_1 w_2) w_3 = w_1 (w_2 w_3)$, where each w_i is either u or belongs to V . If $w_1 = w_2 = w_3 = u$, then $(uu)u = \alpha u = u(uu)$. If at least two of w_i 's belong to V , then $(w_1 w_2) w_3 = 0 = w_1 (w_2 w_3)$. For the remaining three cases, let $v \in V$: then $(uu)v = \alpha v = \varphi^2 v = \varphi(\varphi v) = u(uv)$, and $(uv)u = -\varphi(\varphi v) = \varphi(-\varphi v) = u(vu)$, and $(vu)u = -\varphi(-\varphi v) = \varphi^2 v = \alpha v = v(uu)$.

In the forthcoming classification of quadratic extension algebras we distinguish two cases, depending on whether α is a square or not.

CASE 1: $\alpha = \delta^2 \in \square K^\times$.

Let us say that a K -algebra is of **type 1** if it is a quadratic extension K -algebra of rank 2 that satisfies the condition of this case. (Recall that $\alpha \cdot \square K^\times$ is determined by a quadratic extension algebra of rank 2.) Every K -algebra isomorphic to a K -algebra of type 1 is itself of type 1.

Replacing u by u/δ , we can assume that $u^2 = 1$, and hence that $\varphi^2 = \text{id}_V$. The linear transformations $\pi_+ := \frac{1}{2}(\text{id}_V + \varphi)$ and $\pi_- := \frac{1}{2}(\text{id}_V - \varphi)$ of V are complementary projectors onto the invariant subspaces $V_+ := \pi_+ V$ and $V_- := \pi_- V$ of φ , so we have $V = V_+ \oplus V_-$. Since $\varphi\pi_+ = \pi_+$ and $\varphi\pi_- = -\pi_-$, the linear transformation φ of V restricts on V_+ to id_{V_+} and on V_- to $-\text{id}_{V_-}$, that is, $\varphi v = v$ for every $v \in V_+$ and $\varphi v = -v$ for every $v \in V_-$. Set $m_+ := \dim V_+$ and $m_- := \dim V_-$. We can always assume that $m_+ \geq m_-$: if it happens that $m_+ < m_-$, we simply replace u by $-u$. The pair of cardinal numbers (m_+, m_-) with $m_+ \geq m_-$ is completely determined by A (we shall say that the pair and A are **associated**), because A determines the (unordered) pair of subspaces $\{V_+, V_-\}$: if u' is any pure element of A such that $(u')^2 = 1$, then $u' = \pm u + v$ for some $v \in V$, and we have $\{(1+u') \text{ rad } \vec{A}, (1-u') \text{ rad } \vec{A}\} = \{V_+, V_-\}$.

It is clear that two K -algebras of type 1 are isomorphic if and only if the pairs of cardinal numbers associated with them are equal.

Moreover, every pair of cardinal numbers (m_+, m_-) , $m_+ \geq m_-$, is associated with some K -algebras of type 1, that is, it determines an isomorphism class of K -algebras. Given (m_+, m_-) , we define a multiplication on $A := K \oplus K \oplus K^{(m_+)} \oplus K^{(m_-)}$ by

$$\begin{aligned} (\tau_1, \xi_1, v_1, v'_1) \cdot (\tau_2, \xi_2, v_2, v'_2) = \\ \left(\tau_1\tau_2 + \xi_1\xi_2, \tau_1\xi_2 + \tau_2\xi_1, \tau_1v_2 + \tau_2v_1 + \xi_1v_2 - \xi_2v_1, \tau_1v'_2 + \tau_2v'_1 - \xi_1v'_2 + \xi_2v'_1 \right). \end{aligned}$$

Then A is a K -algebra of type 1 associated with the pair (m_+, m_-) .

Examples. The K -algebra $\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ is of type 1, associated with $(1, 0)$. The K -algebra $\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$ is of type 1, associated with $(1, 1)$.

CASE 2: $\alpha \in K^\times \setminus \square K^\times$.

We shall say that a K -algebra is of **type 2** if it is a quadratic extension K -algebra of rank 2 satisfying the condition of this case.

Let \mathcal{U} be the set of all subspaces U of the K -space V such that $U \cap \varphi U = 0$, partially ordered by inclusion. It is easy to see that the union of a nonempty chain of subspaces belonging to \mathcal{U} is a subspace belonging to \mathcal{U} , thus \mathcal{U} possesses maximal elements.

Let $U \in \mathcal{U}$ be such that $W := U + \varphi U \neq V$; then U is not maximal in \mathcal{U} . In order to prove this, first note that W is an invariant subspace of φ (because $\varphi(\varphi U) = \alpha U = U$), then choose any $v \in V \setminus W$; we claim that $\varphi v \notin W + Kv = W \oplus Kv$. Suppose, to the contrary, that $\varphi v = w + \xi v$ for some $w \in W$ and some $\xi \in K$; then $\alpha v = \varphi^2 v = \varphi w + \xi \varphi v = (\varphi w + \xi w) + \xi^2 v$ implies $\alpha v = \xi^2 v$, hence $\alpha = \xi^2$, a contradiction. We have a direct sum $U \oplus \varphi U \oplus Kv \oplus K\varphi v$ of subspaces of V , and it follows that $U \oplus Kv \in \mathcal{U}$.

Pick a maximal $U \in \mathcal{U}$; then $V = U \oplus \varphi U$. Define the K -isomorphism ψ from $U \oplus U$ (an external direct sum) to V by $\psi(v, v') = v + \varphi v'$, and define the K -automorphism φ' of $U \oplus U$ by $\varphi'(v, v') = (\alpha v', v)$. Then $\psi\varphi'(v, v') = \varphi v + \alpha v' = \varphi\psi(v, v')$, thus ψ is an isomorphism $(U \oplus U, \varphi') \rightarrow (V, \varphi)$ of K -spaces endowed with linear transformations.

The dimension m of a maximal subspace U is determined by A : if $\dim A$ is finite, then $m = \frac{1}{2} \dim A - 1$, and if $\dim A$ is infinite, then $m = \dim A$. An isomorphism class of K -algebras of type 2 is determined by a pair $(\bar{\alpha}, m)$, where $\bar{\alpha}$ is a nontrivial element of the group $K^\times / \square K^\times$ and m is a cardinal number. A construction of a K -algebra of type 2, given any such pair $(\bar{\alpha}, m)$, where $\bar{\alpha} = \alpha \cdot \square K^\times$ for some non-square $\alpha \in K^\times$: we equip the vector K -space $A := K \oplus K \oplus K^{(m)} \oplus K^{(m)}$ with the multiplication

$$\begin{aligned} (\tau_1, \xi_1, v_1, v'_1) \cdot (\tau_2, \xi_2, v_2, v'_2) = \\ \left(\tau_1 \tau_2 + \alpha \xi_1 \xi_2, \tau_1 \xi_2 + \tau_2 \xi_1, \tau_1 v_2 + \tau_2 v_1 + \alpha \xi_1 v'_2 - \alpha \xi_2 v'_1, \tau_1 v'_2 + \tau_2 v'_1 + \xi_1 v_2 - \xi_2 v_1 \right); \end{aligned}$$

this A is a K -algebra of type 2 belonging to the isomorphism class determined by $(\bar{\alpha}, m)$.

Example. The K -algebra $\left(\frac{\alpha, 0}{K}\right)$, α a non-square, is of type 2, associated with $(\bar{\alpha}, 1)$.

11 Some characterizations of quaternion algebras

Let A be a quadratic extension K -algebra of rank at least 3.

Since $\text{rank } \vec{A} \geq 2$, there exists a two-dimensional regular subspace U of the quadratic space (\vec{A}, ν_A) . We can choose an orthogonal basis (e, f) of U ; then $a := e^2 = -\langle e, e \rangle$ and $b := f^2 = -\langle f, f \rangle$ are non-zero scalars, and $\langle e, f \rangle = 0$ hence $ef = e \times f$ is pure. We have

$$\langle e, f, ef \rangle = \langle e \times f, e \times f \rangle = -(e \times f)^2 = e^2 f^2 - \langle e, f \rangle^2 = ab \neq 0,$$

thus, according to Lemma 13, the pure elements e, f, ef form a basis of the K -space \vec{A} , and we see that A is isomorphic to the quaternion algebra $\left(\frac{a, b}{K}\right)$.

Recall that, given any $a, b \in K^\bullet$, one way to define the quaternion algebra $\left(\frac{a, b}{K}\right)$ is as the K -algebra with the K -basis $(1, i, j, k)$, where $i^2 = a$, $j^2 = b$, $ij = k$ and $ji = -k$. Assuming associativity of multiplication, we derive the rest of the multiplication table, namely $k^2 = -ji \cdot ij = -ji^2j = -i^2j^2 = -ab$, $ik = i \cdot ij = i^2j = aj$, $ki = -ji \cdot i = -aj$, and similarly $jk = -bi$, $kj = bi$; direct calculations then show that the multiplication determined by the full multiplication table is indeed associative.¹⁰ A quaternion K -algebra is defined as any K -algebra isomorphic to $\left(\frac{a, b}{K}\right)$ for some $a, b \in K^\bullet$. In other words, Q is a quaternion K -algebra if and only if Q is a K -algebra, there exist non-zero scalars a and b , and there exists a basis $(1, e, f, g)$ of the K -space Q , so that $e^2 = a$, $f^2 = b$, and $ef = -fe = g$; we shall call any such basis an (a, b) -basis of Q ;¹¹

¹⁰We also obtain an associative algebra if one of the scalars a and b is 0, or even if both of them are 0, though then $\left(\frac{a, b}{K}\right)$ is not a quaternion algebra—it is just a quaternionish algebra.

¹¹Mark that a quaternion algebra may have many (a, b) -bases for particular scalars $a, b \in K^\bullet$, and that it may have (a, b) -bases for different pairs of scalars (a, b) . To give a simple example for the latter, if a quaternion algebra has an (a, b) -basis, then it has an $(\lambda^2 a, \mu^2 b)$ -basis for any non-zero scalars λ and μ .

also, we shall call (e, f, g) an (a, b) -basis of \vec{Q} . A quaternion algebra Q with an (a, b) -basis $(1, e, f, g)$ is a quadratic extension algebra with $\vec{Q} = Ke + Kf + Kg$, because $(\xi e + \eta f + \zeta g)^2 = a\xi^2 + b\eta^2 - ab\zeta^2$ is a scalar for any scalars ξ, η , and ζ .

Lemma 19. *Suppose that a non-zero K -algebra Q is generated by elements e and f such that $e^2 = a$ and $f^2 = b$ are non-zero scalars and $ef = -fe$. Then Q is a quaternion K -algebra, and $(1, e, f, ef)$ is its (a, b) -basis.*

Proof. Since Q is a non-zero algebra, it contains the field of scalars $K = K1$. It is clear that the K -space Q is spanned by $1, e, f$, and $g := ef$, so it remains to prove that these four elements are linearly independent. Given an invertible $y \in Q$ we define a linear transformation κ_y of Q by $\kappa_y x := \frac{1}{2}(x + yxy^{-1})$ for $x \in Q$. The elements e and f are invertible ($e^{-1} = a^{-1}e, f^{-1} = b^{-1}f$), so we have the linear transformations κ_e and κ_f . Since $e1e^{-1} = 1, eee^{-1} = e, efe^{-1} = -fee^{-1} = -f, ege^{-1} = e(-fe)e^{-1} = -ef = -g$, we have $\kappa_e 1 = 1, \kappa_e e = e$, and $\kappa_e f = \kappa_e g = 0$, and similarly $\kappa_f 1 = 1, \kappa_f f = f$, and $\kappa_f e = \kappa_f g = 0$. Let $x = \alpha + \xi e + \eta f + \zeta g \in Q$ (where $\alpha, \xi, \eta, \zeta \in K$); then $\kappa_e x = \alpha + \xi e$ and $\kappa_f x = \alpha + \eta f$, hence $\kappa_e \kappa_f x = \alpha$. Setting $\tau := \kappa_e \kappa_f$ we have $\tau x = \alpha, \tau(ex) = a\xi, \tau(fx) = b\eta$, and $\tau(gx) = -ab\zeta$. But then $x = 0$ implies $\alpha = \xi = \eta = \zeta = 0$, which means that $1, e, f, g$ are linearly independent. \square

Proposition 20. *Every quaternion K -algebra is a simple algebra with center K .*

Proof. Let Q be a quaternion K -algebra. There exist $a, b \in K^\bullet$ such that Q has an (a, b) -basis $(1, e, f, g)$. If the element $x = \alpha + \xi e + \eta f + \zeta g$ of Q belongs to $Z(Q)$, then comparing the coefficients in $ex = xe$ we find that $\eta = \zeta = 0$, and similarly $fx = xf$ gives us $\xi = \zeta = 0$, therefore $x = \alpha \in K$.

Let $J \neq Q$ be an ideal of Q , and let $Q \rightarrow Q/J : x \mapsto \bar{x}$ be the natural projection. The elements $\bar{1}, \bar{e}, \bar{f}, \bar{g}$ of the non-zero K -algebra Q/J obey the same multiplication table as the elements $1, e, f, g$ of Q , thus, by Lemma 19, Q/J is a quaternion K -algebra with an (a, b) -basis $(\bar{1}, \bar{e}, \bar{f}, \bar{g})$, and it follows that $J = 0$. \square

We shall need the following fact, borrowed from the theory of central simple algebras (in order to prove the implication (3) \implies (2) in Proposition 21 below): every element x of a four-dimensional central simple K -algebra is a zero of its reduced characteristic polynomial $X^2 - t(x)X + n(x) \in K[X]$, where $t(x) \in K$ is the reduced trace of x and $n(x) \in K$ is the reduced norm of x .

Now it is easy to derive, from the foregoing results, the two highlighted results in the preceding two sections, and the overview of algebras of dimension at most 3 in Section 5, the following characterizations of quaternion algebras (we omit the evident proofs):

Proposition 21. *The following properties of a K -algebra A are equivalent:*

- (1) A is a quaternion algebra;
- (2) $A \neq K$ is a simple quadratic extension algebra with center K ;
- (3) $A \neq K$ is a simple algebra with center K and $\dim_K A \leq 4$;
- (4) $A = K \oplus V$ is a ground Clifford algebra and $\text{rank}(V, \square_V) \geq 2$;
- (5) $A = K \oplus V$ is a ground Clifford algebra of dimension at least 3 and the quadratic space (V, \square_V) is regular.

12 (Anti)automorphisms of a quaternion algebra

To begin with, we consider properties of an (anti)automorphism h of an arbitrary quadratic extension algebra A .

If $\alpha \in K$, then $h(\alpha) = h(\alpha 1) = \alpha h(1) = \alpha$, that is, h fixes all scalars. Clearly h preserves squares (whether it is anti- or not): that is, $h(x^2) = h(x)^2$ for every $x \in A$. For every $u \in A \setminus K$ we have $h(u) \in A \setminus K$ because h is a bijection that fixes all scalars. If $u \in \vec{A}$, then $h(u) \in A \setminus K$ and $h(u)^2 = h(u^2) = u^2 \in K$, thus $h(u) \in \vec{A}$; since also $h^{-1}(v) \in \vec{A}$ for every $v \in \vec{A}$, we see that h restricts to an automorphism \vec{h} of the K -space \vec{A} . The K -linear transformation \vec{h} preserves the scalar product $\langle -, - \rangle$ on \vec{A} : for all $u, v \in \vec{A}$, $\langle \vec{h}u, \vec{h}v \rangle = -\frac{1}{2}(h(u)h(v) + h(v)h(u)) = h(-\frac{1}{2}(uv + vu)) = h(\langle u, v \rangle) = \langle u, v \rangle$. (This is also a straightforward consequence of the fact that h preserves all squares.) In other words, \vec{h} is a bijective isometry of the quadratic space (\vec{A}, ν_A) , i.e., it belongs to the orthogonal group $O(\nu_A)$.

We denote by $\text{Aut}(A)$ the group of all automorphisms of the K -algebra A , and by $\overline{\text{Aut}}(A)$ the group consisting of all automorphisms and of all antiautomorphisms of the K -algebra A . Let us allow writing the conjugation on A as an in-line “ $*$ ”, so that $*x$ is an alternative notation for x^* , for any $x \in A$. Every $h \in \overline{\text{Aut}}(A)$ commutes with the conjugation: $h* = *h$, that is, $h(x^*) = h(x)^*$ for every $x \in A$. Indeed, if $x = \alpha + u$ ($\alpha \in K$, $u \in \vec{A}$), then $h(x) = \alpha + h(u)$ with $h(u) \in \vec{A}$, and $h(x^*) = \alpha - h(u) = (\alpha + h(u))^* = h(x)^*$. We write $h^* := h* = *h$ for every $h \in \overline{\text{Aut}}(A)$, and denote by $\text{Aut}(A)^*$ the set of all antiautomorphisms of the K -algebra A . If $h \in \text{Aut}(A)$, then $h^* \in \text{Aut}(A)^*$, and if $h \in \text{Aut}(A)^*$, then $h^* \in \text{Aut}(A)$.

When A is commutative (there are commutative quadratic extension algebras), every antiautomorphism of A is in fact an automorphism of A , and hence $\overline{\text{Aut}}(A) = \text{Aut}(A)$. But, if A is not commutative, then no antiautomorphism of A is an automorphism of A , (in particular, the conjugation is not an automorphism), the group $\text{Aut}(A)$ is a normal subgroup of index 2 of the group $\overline{\text{Aut}}(A)$, and $\text{Aut}(A)^*$ is the only nontrivial coset of $\text{Aut}(A)$ in $\overline{\text{Aut}}(A)$.

We can always recover $h \in \overline{\text{Aut}}(A)$ from \vec{h} , as $h = \text{id}_K \oplus \vec{h}$, which means that the homomorphism of groups $\overline{\text{Aut}}(A) \rightarrow \text{O}(\nu_A) : h \mapsto \vec{h}$ is injective.

If h is an automorphism, then \vec{h} preserves the pure products and the mixed products: $\vec{h}(u \times v) = \vec{h}u \times \vec{h}v$ and $\langle \vec{h}u, \vec{h}v, \vec{h}w \rangle = \langle u, v, w \rangle$ for all $u, v, w \in \vec{A}$. If h is an anti-automorphism, then $\vec{h}(u \times v) = -\vec{h}u \times \vec{h}v$ and $\langle \vec{h}u, \vec{h}v, \vec{h}w \rangle = -\langle u, v, w \rangle$ for $u, v, w \in \vec{A}$.

Let Q be a quaternion K -algebra. (Recall that Q is not commutative.)

Now the mixed product is a non-zero alternating trilinear form on \vec{Q} . If φ is a linear transformation of \vec{Q} (that is, an endomorphism of the K -space \vec{Q}), then $\det \varphi$ is the unique scalar δ such that $\langle \varphi u, \varphi v, \varphi w \rangle = \delta \cdot \langle u, v, w \rangle$ for all $u, v, w \in \vec{Q}$. The determinant of an isometry of the regular quadratic space (\vec{Q}, ν_Q) can only be 1 or -1 . The special orthogonal group $\text{SO}(\nu_Q) = \text{O}_+(\nu_Q)$ is defined as the subgroup of $\text{O}(\nu_Q)$ consisting of all $\sigma \in \text{O}(\nu_Q)$ with $\det \sigma = 1$. We write the set of all $\sigma \in \text{O}(\nu_Q)$ with $\det \sigma = -1$ as $\text{O}_-(\nu_Q)$. The subgroup $\text{SO}(\nu_Q)$ of $\text{O}(\nu_Q)$ is normal of index 2, and $\text{O}_-(\nu_Q) = -\text{SO}(\nu_Q)$ is the only nontrivial coset of $\text{SO}(\nu_Q)$ in $\text{O}(\nu_Q)$.

If $h \in \text{Aut}(Q)$, then $\langle \vec{h}u, \vec{h}v, \vec{h}w \rangle = \langle u, v, w \rangle$ for all $u, v, w \in \vec{Q}$, thus $\det \vec{h} = 1$, and if $h \in \text{Aut}(Q)^*$, then $\det \vec{h} = -1$. The group homomorphism $\overline{\text{Aut}}(Q) \rightarrow \text{O}(\nu_Q) : h \mapsto \vec{h}$ restricts to a group homomorphism $\text{Aut}(Q) \rightarrow \text{SO}(\nu_Q)$. We already know that both these group homomorphisms are injective. They are in fact group isomorphisms:

Proposition 22. *Let Q be a quaternion K -algebra, and let $\sigma \in \text{O}(\nu_Q)$. If $\sigma \in \text{O}_+(\nu_Q)$, then $\text{id}_K \oplus \sigma \in \text{Aut}(Q)$, and if $\sigma \in \text{O}_-(\nu_Q)$, then $\text{id}_K \oplus \sigma \in \text{Aut}(Q)^*$.*

Proof. Let $\sigma \in \text{O}_+(\nu_Q)$. There exists an (a, b) -basis (e, f, ef) of \vec{Q} for some $a, b \in K^\bullet$. Since σ preserves scalar products, we have $(\sigma e)^2 = -\langle \sigma e, \sigma e \rangle = -\langle e, e \rangle = e^2 = a$, and likewise $(\sigma f)^2 = f^2 = b$; also $\sigma e \perp \sigma f$, because $\langle \sigma e, \sigma f \rangle = \langle e, f \rangle = 0$. It follows that $(\sigma e, \sigma f, (\sigma e)(\sigma f))$ is an (a, b) -basis of \vec{Q} . The non-zero vectors $\sigma(e f)$ and $(\sigma e)(\sigma f)$ both lie in the one-dimensional subspace $\{\sigma e, \sigma f\}^\perp$ of \vec{Q} , so are proportional. Since $\langle \sigma e, \sigma f, \sigma(e f) \rangle = \det(\sigma) \langle e, f, ef \rangle = ab$ and $\langle \sigma e, \sigma f, (\sigma e)(\sigma f) \rangle = ab$, the proportional vectors $\sigma(e f)$ and $(\sigma e)(\sigma f)$ must be equal, so $(1, \sigma e, \sigma f, \sigma(e f)) = (1, \sigma e, \sigma f, (\sigma e)(\sigma f))$ is an (a, b) -basis of Q , and we conclude that $\text{id}_K \oplus \sigma$ is an automorphism of Q .

Let $\sigma \in \text{O}_-(\nu_Q)$. Since $-\sigma$ preserves scalar products and $\det(-\sigma) = (-1)^3 \det(\sigma) = 1$, we have $-\sigma \in \text{O}_+(\nu_Q)$, thus $h := \text{id}_K \oplus -\sigma \in \text{Aut}(Q)$ by what we have proved above, and it follows that $h^* = *h = \text{id}_K \oplus \sigma \in \text{Aut}(Q)^*$. \square

Corollary 23. *For every quaternion K -algebra Q the mapping $\overline{\text{Aut}}(Q) \rightarrow \text{O}(\nu_Q) : h \mapsto \vec{h}$ is a group isomorphism that restricts to the isomorphism $\text{Aut}(Q) \rightarrow \text{SO}(\nu_Q)$.*

For every invertible $y \in Q$ we have the inner automorphism $c_y: Q \rightarrow Q : x \mapsto yxy^{-1}$ of the K -algebra Q . The mapping $c: Q^\times \rightarrow \text{Aut}(Q) : y \mapsto c_y$ is a homomorphism of groups. Below we shall show that the homomorphism c is surjective and that $\ker c = K^\times$.

Let w be an anisotropic vector of the quadratic space (\vec{Q}, ν_Q) . Then $\vec{Q} = Kw \perp w^\perp$, where w^\perp is a two-dimensional hyperplane of \vec{Q} . The linear transformation ϱ_w of \vec{Q} , defined by $\varrho_w u := -wuw^{-1}$ for $u \in \vec{Q}$, is the reflection along the vector w across the hyperplane w^\perp ; indeed, clearly $\varrho_w w = -w$, and if $u \in w^\perp$, then $\varrho_w u = (-wu)w^{-1} = uww^{-1} = u$. The reflection ϱ_w is an isometry of (\vec{Q}, ν_Q) of determinant -1 , and $\text{id}_K \oplus \varrho_w$ is the antiautomorphism $*c_w = c_w^*$ of Q .

Lemma 24. *Every automorphism of a quaternion K -algebra Q is inner.*

Proof. Let $h \in \text{Aut}(Q)$; then $\vec{h} \in \text{O}_+(\nu_Q)$. Since id_Q is certainly an inner automorphism of Q , we can assume that $\vec{h} \neq \text{id}_{\vec{Q}}$. According to the Cartan-Dieudonné theorem, \vec{h} is a composite of two reflections (since $\det \vec{h} = 1$, \vec{h} cannot be a single reflection or a composite of three reflections), thus there exist two anisotropic vectors $v, w \in \vec{Q}$ such that $\vec{h} = \varrho_v \varrho_w$. But then $h = \text{id}_K \oplus \varrho_v \varrho_w = c_v^* c_w^* = c_v ** c_w = c_v c_w = c_{vw}$. \square

Proposition 25. *If Q is a quaternion K -algebra, then $\text{Aut}(Q) \cong Q^\times / K^\times$.*

Proof. The group homomorphism $c: Q^\times \rightarrow \text{Aut}(Q)$ is surjective, in view of Lemma 24. It is clear that the kernel of c contains K^\times . Conversely, if $y \in \ker c$, then certainly $y \neq 0$, and $yx = xy$ for every $x \in Q$ hence $y \in Z(Q) = K$. We conclude that $\ker c = K^\times$, and that the surjective homomorphism c induces an isomorphism $Q^\times / K^\times \rightarrow \text{Aut}(Q)$. \square

13 Orthogonal bases of pure quaternions

Let Q be a quaternion K -algebra with an (a, b) -basis $(1, e, f, ef)$. Setting $g := ef$, the full multiplication table for the pure basis elements e, f, g is

$$\begin{aligned} e^2 &= a, & f^2 &= b, & g^2 &= -ab, \\ ef &= -fe = g, & fg &= -gf = -be, & ge &= -eg = -af. \end{aligned}$$

This multiplication table is somewhat lopsided; it is well suited for some purposes since it involves only two scalars a and b , but it definitely lacks symmetry.

In order to bring about the desired symmetry, we consider an arbitrary orthogonal basis (e, f, g) of \vec{Q} . Writing $a := e^2 \in K^\bullet$ and $b := f^2 \in K^\bullet$, we note that $(1, e, f, ef)$ is an (a, b) -basis of Q . The orthogonal complement of $\{e, f\}$ in \vec{Q} is a one-dimensional

subspace of \vec{Q} which contains both $ef \neq 0$ and $g \neq 0$, thus $ef = \gamma g$ for some $\gamma \in K^\bullet$; it is clear that γ may be any non-zero scalar. Now $fg = -b\gamma^{-1}e$ and $ge = -a\gamma^{-1}f$, which suggests that we set $\alpha := -b\gamma^{-1}$ and $\beta := -a\gamma^{-1}$, and this indeed makes the multiplication table symmetric:

$$\begin{aligned} e^2 &= -\beta\gamma, & f^2 &= -\alpha\gamma, & g^2 &= -\alpha\beta, \\ ef &= -fe = \gamma g, & fg &= -gf = \alpha e, & ge &= -eg = \beta f. \end{aligned} \tag{11}$$

We shall call such a basis $(1, e, f, g)$ of a quaternion algebra Q an (α, β, γ) -**basis of Q** . Note that the multiplication rules $e^2 = -\beta\gamma$, $f^2 = -\alpha\gamma$, $ef = -fe = \gamma g$ imply all other entries of the multiplication table (which is not at all surprising, of course); for example, from $\gamma^2 g^2 = ef \cdot ef = -fe \cdot ef = -e^2 f^2 = -\alpha\beta\gamma^2$ we get $g^2 = -\alpha\beta$, while $\gamma fg = f \cdot ef = -f \cdot fe = -f^2 e = \alpha\gamma e$ gives us $fg = \alpha e$. Let us compute the mixed product $\langle e, f, g \rangle$:

$$\langle e, f, g \rangle = \langle ef, g \rangle = \langle \gamma g, g \rangle = -\gamma g^2 = \alpha\beta\gamma.$$

The symmetry persists.

Given $\alpha, \beta, \gamma \in K^\bullet$, we let the quaternion K -algebra $Q_K(\alpha, \beta, \gamma)$ be the K -algebra with the basis $(1, i, j, k)$ that satisfies the multiplication table (11) (with (i, j, k) in place of (e, f, g) , of course). There is really no need to prove associativity of the K -algebra $Q_K(\alpha, \beta, \gamma)$, because we can construct it as the K -algebra $(\frac{-\beta\gamma, -\alpha\gamma}{K})$ in which we then choose the (α, β, γ) -basis $(1, i, j, \gamma^{-1}ij)$. However, a reader having a streak of mathematical masochism in him/her is going to enjoy the following verification (by raw brute force) of the associativity of the K -algebra $Q = Q_K(\alpha, \beta, \gamma)$. Choosing Q to be K^4 , a K -space with the standard basis $1_Q = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$, $k = (0, 0, 0, 1)$, we define the (bilinear) multiplication on Q by

$$\begin{aligned} (\tau_1, \xi_1, \eta_1, \zeta_1) (\tau_2, \xi_2, \eta_2, \zeta_2) &:= (\tau_1\tau_2 - \beta\gamma\xi_1\xi_2 - \alpha\gamma\eta_1\eta_2 - \alpha\beta\zeta_1\zeta_2, \\ &\tau_1\xi_2 + \tau_2\xi_1 + \alpha\eta_1\zeta_2 - \alpha\zeta_1\eta_2, \\ &\tau_1\eta_2 + \tau_2\eta_1 + \beta\zeta_1\xi_2 - \beta\xi_1\zeta_2, \\ &\tau_1\zeta_2 + \tau_2\zeta_1 + \gamma\xi_1\eta_2 - \gamma\eta_1\xi_2). \end{aligned}$$

Now we consider three arbitrary elements $x_i := (\tau_i, \xi_i, \eta_i, \zeta_i)$ ($i = 1, 2, 3$) of Q , and compare $(x_1x_2)x_3$ with $x_1(x_2x_3)$; we obtain a quartet of identities in the polynomial ring $\mathbb{Z}[\alpha, \beta, \gamma, \tau_1, \xi_1, \eta_1, \zeta_1, \tau_2, \xi_2, \eta_2, \zeta_2, \tau_3, \xi_3, \eta_3, \zeta_3]$, where for the few moments it takes to make the comparison we regard $\alpha, \beta, \gamma, \tau_1, \xi_1, \dots, \eta_3, \zeta_3$ as distinct formal variables.¹²

¹²We do not carry out the comparison ourselves—manual calculations are too prone to mistakes—instead we hand the task over to a computer program, such as MATHEMATICA, which is better at handling symbolic expressions than we are.

This is how the constructions $\left(\frac{a,b}{K}\right)$ and $Q_K(\alpha, \beta, \gamma)$ of quaternion algebras are related:

$$Q_K(\alpha, \beta, \gamma) \cong \left(\frac{-\beta\gamma, -\alpha\gamma}{K}\right)$$

for all $\alpha, \beta, \gamma \in K^\bullet$, and, in the other direction,

$$\left(\frac{a,b}{K}\right) \cong Q_K(-b, -a, 1)$$

for all $a, b \in K^\bullet$.

14 Arbitrary bases of pure quaternions

Let Q be a quaternion K -algebra. This time we write the pure product of the elements u, v of \vec{Q} as $u \times_Q v$. Let (v_1, v_2, v_3) be an arbitrary basis of \vec{Q} , and set $w_1 := v_2 \times_Q v_3$, $w_2 := v_3 \times_Q v_1$, $w_3 := v_1 \times_Q v_2$; we know that (w_1, w_2, w_3) is a basis of \vec{Q} .

Let us regard the elements of the K -space K^3 as columns; that is, $(x_1, x_2, x_3) \in K^3$ is just an in-line notation for the column $[x_1 \ x_2 \ x_3]^\top$. For $x, y \in K^3$ we let $x \times y$ denote the usual cross product of triples. We define isomorphisms of K -spaces $\varphi, \psi: K^3 \rightarrow \vec{Q}$ by

$$\varphi x := x_1 v_1 + x_2 v_2 + x_3 v_3, \quad \psi x := x_1 w_1 + x_2 w_2 + x_3 w_3,$$

for every $x = (x_1, x_2, x_3) \in K^3$. Expanding vectors w_1, w_2, w_3 in the basis (v_1, v_2, v_3) as $w_j = \sum_{i=1}^3 \alpha_{ij} v_i$ ($j = 1, 2, 3$) and denoting the matrix $[\alpha_{ij}]_{3 \times 3}$ by M , we have

$$\psi x = \varphi Mx \quad \text{for every } x \in K^3.$$

Let $M' := [\langle v_i, v_j \rangle]_{3 \times 3}$. Then $\langle \varphi x, \varphi y \rangle = x^\top M' y$ for all $x, y \in K^3$.

Let $d := \langle v_1, v_2, v_3 \rangle$; then $\det M' = d^2$, by the ‘grand identity’. The nine identities

$$d\delta_{ij} = \langle v_i, w_j \rangle = \sum_{k=1}^3 \langle v_i, v_k \rangle \alpha_{kj}, \quad i, j = 1, 2, 3,$$

can be rewritten as a single identity $M' M = dI$, and we see that $M = d \cdot (M')^{-1}$ is an invertible symmetric matrix. From $\det(M') \det(M) = \det(dI) = d^3$ we get $\det M = d$, whence $M' = dM^{-1} = \tilde{M}$ (where \tilde{M} is the adjugate of the matrix M), thus

$$\langle \varphi x, \varphi y \rangle = x^\top \tilde{M} y \quad \text{for all } x, y \in K^3;$$

knowing the matrix M , we of course know the adjugate matrix \tilde{M} , and we can compute the scalar product of the pure quaternions φx and φy . We can also compute the pure product, since

$$\varphi x \times_Q \varphi y = \psi(x \times y) = \varphi(M(x \times y)) \quad \text{for all } x, y \in K^3.$$

With the formulas for the scalar product and the pure product in our hands we can now write out the formula for the product of general quaternions,

$$(s + \varphi x)(t + \varphi y) = (st - x^\top \tilde{M}y) + \varphi(sy + tx + M(x \times y)) ,$$

where $s, t \in K$ and $x, y \in K^3$.

For every nonsingular symmetric 3×3 matrix M with entries in K there exists a quaternion K -algebra Q with a basis (v_1, v_2, v_3) of \vec{Q} such that $\psi = \varphi M$, where $\varphi, \psi: K^3 \rightarrow \vec{Q}$ are defined as above. Given the matrix M , we construct the quaternion K -algebra $Q = Q_K(M) = K^4 = K \oplus K^3$ with the standard basis (v_1, v_2, v_3) of $K^3 = \vec{Q}$, which has the required property, where the multiplication on Q is defined by

$$(s \oplus x)(t \oplus y) := (st - x^\top \tilde{M}y) \oplus (sy + tx + M(x \times y)) .$$

Associativity of multiplication is easily verified by raw brute force (or via some clever shortcut). Since $(0 \oplus x)^2 = (-x^\top \tilde{M}x) \oplus 0$ for every $x \in K^3$, we see that Q is a ground Clifford algebra. And finally, $\text{rank } \vec{Q} = 3$ because \tilde{M} is nonsingular, thus Q is indeed a quaternion algebra.

The quaternion K -algebra Q with an (α, β, γ) -basis, constructed in the preceding section, is a special case: $Q_K(\alpha, \beta, \gamma) = Q_K(\text{diag}(\alpha, \beta, \gamma))$.

15 Quaternion algebras over integral domains

This section offers a glimpse of the quaternion algebras over integral domains (of characteristic not 2).

Why under the sun would anyone want to observe a quaternion algebra Q from a viewpoint of an arbitrary basis of \vec{Q} ? There always exist orthogonal bases for the norm ν_Q , and with respect to any such basis the matrix of the associated scalar product is diagonal, which facilitates the exploration of the quaternion algebra. However, we are able to diagonalize because the algebra Q is over a *field* K , so that the quadratic functional ν_Q is defined on the *vector space* \vec{Q} . What if we came across a quaternion algebra defined over an integral domain (of characteristic not 2)? Then it is no longer true that we can diagonalize at will if a quaternion algebra is presented relative to a general basis; when we do happen to find an orthogonal basis, this appears to be nothing short of a miracle and is always an occasion (well, a good excuse) for celebration.

The quaternion algebras over integral domains we are talking about are constructed as follows. Let R be an integral domain of characteristic different from 2, and let M

be a symmetric nonsingular 3×3 matrix with entries in R .¹³ The quaternion algebra $Q = Q_R(M)$ is defined as the free R -module $R \oplus R^3$ (whose elements we write as plain sums $r + u$, where $r \in R$ and $u \in R^3$)¹⁴ with the R -bilinear multiplication defined by¹⁵

$$(r + u)(s + v) := (rs - u^\top \tilde{M}v) + (rv + su + M(u \times v)) , \quad (12)$$

for all $r + u, s + v \in Q$. We denote the vectors of the standard basis of R^3 by e_1, e_2, e_3 , so that the general element of Q is $s + u = s + u_1e_1 + u_2e_2 + u_3e_3$, and write $\vec{Q} := R^3$.

Let K be the field of fractions of R . The R -algebra structure $Q = Q_R(M)$ on $R \oplus R^3$ uniquely extends to the K -algebra structure $Q_K = Q_K(M)$ on $K \oplus K^3$,¹⁶ in which the multiplication is defined by the same formula (12) as the multiplication in Q , only that now $r + u, s + v \in Q_K$. The standard basis $(1, e_1, e_2, e_3)$ of the free R -module Q is also a basis of the K -space Q_K . Since Q_K is a bona fide quaternion algebra over the field K , it makes perfect sense to call Q a quaternion algebra over the integral domain R .

We still have, on the three-dimensional free R -module $\vec{Q} = R^3$, the scalar product $\langle u, v \rangle = u^\top \tilde{M}v$ (with values in R), the pure product $u \times_Q v = M(u \times v)$, and the mixed product $\langle u, v, w \rangle$. Also there is the conjugation, which is an antiautomorphism of Q , and there are the norm $N: Q \rightarrow R$ and its restriction $\nu_Q: \vec{Q} \rightarrow R$.

Every automorphism h of the R -algebra Q extends to a unique automorphism h_K of the K -algebra Q_K ; h_K has the same matrix as h with respect to the standard basis, and $h_K(Q) = Q$. Conversely, if h is an automorphism of Q_K whose matrix with respect to the standard basis has all entries in R then h restricts to the automorphism h_R of Q ; indeed, since $\det h = 1$, the matrix of h is invertible in $\mathbb{M}_4(R)$, thus the restriction h_R is an automorphism of the R -module Q and hence of the R -algebra Q .

If h is an automorphism of the R -algebra Q , then its restriction \vec{h} to \vec{Q} is an automorphism of the quadratic R -module (\vec{Q}, ν_Q) with $\det \vec{h} = 1$, that is, $\vec{h} \in \text{SO}(\nu_Q)$. Conversely, let $\sigma \in \text{SO}(\nu_Q)$; then $\text{id}_R \oplus \sigma \in \text{Aut}(Q)$. To prove this claim, we extend σ to the automorphism σ_K of the quadratic K -space (\vec{Q}_K, ν_{Q_K}) ; we know that $h = \text{id}_K \oplus \sigma_K$ is an automorphism of the K -algebra Q_K , and it is clear that the matrix of h has all entries in R , thus the restriction $h_R = \text{id}_R \oplus \sigma$ is an automorphism of the R -algebra Q . It follows that the mapping $\text{Aut}(Q) \rightarrow \text{SO}(\nu_Q) : h \mapsto \vec{h}$ is an isomorphism of groups, with the inverse $\sigma \mapsto \text{id}_R \oplus \sigma$.

Every automorphism h of the K -algebra Q_K is inner: there exists an invertible w in Q_K such that $h(x) = wxw^{-1}$ for every $x \in Q_K$. Rescaling w by a nonzero scalar in K yields the same automorphism h ; since K is the field of fractions of R , we can assume

¹³Nonsingular M means that $\det M \neq 0$.

¹⁴That is, we fearlessly identify $R \oplus 0$ with R and $0 \oplus R^3$ with R^3 .

¹⁵This is the third — and the last — time we write out the blasted formula for multiplication.

¹⁶This is in effect the change of the base ring R to the base field K : $Q_K \cong K \otimes_R Q$.

that $w \in Q$. Then $N(w) = ww^* \in R^\bullet$ and $h(x) = wxw^*/N(w)$, and the matrix W of the K -linear transformation $x \mapsto wxw^*$ of Q_K has all entries in R . It follows that h induces an automorphism of the R -algebra Q if and only if all entries of the matrix W are divisible by $N(w)$. But the matrix W is of the form $W = N(w) \oplus T$, where T is the matrix (relative to the standard basis (e_1, e_2, e_3) of Q_K) of the K -linear transformation \vec{h} of \vec{Q}_K , therefore h induces an automorphism of Q if and only if every entry of the matrix T is divisible by $N(w)$.

Recall that $Q = Q_R(M)$ and $Q_K = Q_K(M)$, where $M \in \mathbb{M}_3(R)$ is a nonsingular symmetric matrix which we write as

$$M = \begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix},$$

where the a_i and the b_j are elements of R and

$$H := \det M = a_1a_2a_3 + 2b_1b_2b_3 - a_1b_1^2 - a_2b_2^2 - a_3b_3^2 \neq 0.$$

We write the adjugate of the matrix M as

$$\tilde{M} = \begin{bmatrix} A_1 & B_3 & B_2 \\ B_3 & A_2 & B_1 \\ B_2 & B_1 & A_3 \end{bmatrix},$$

where $A_1 = a_1a_3 - b_1^2$, $B_3 = -a_3b_3 + b_1b_2$, etc. Let $w = w_0 + w_1e_1 + w_2e_2 + w_3e_3 \in Q$ (the coordinates w_k are in R). The norm of w is

$$N := N(w) = w_0^2 + A_1w_1^2 + A_2w_2^2 + A_3w_3^2 + 2B_3w_1w_2 + 2B_2w_1w_3 + 2B_1w_2w_3,$$

while the entries of the matrix T (defined above) are

$$T_{11} = w_0^2 - 2b_2w_0w_2 + 2b_3w_0w_3 + A_1w_1^2 - A_2w_2^2 - A_3w_3^2 - 2B_1w_2w_3,$$

$$T_{12} = 2b_2w_0w_1 - 2a_1w_0w_3 + 2B_3w_1^2 + 2A_2w_1w_2 + 2B_1w_1w_3,$$

and so on.

Lemma 26. *Let M , H , \tilde{M} , w , N , and T be as in the text. Then:*

$$4w_0^2 = T_{11} + T_{22} + T_{33} + N,$$

$$4Hw_0w_1 = B_2T_{12} - B_3T_{13} + B_1T_{22} - A_2T_{23} + A_3T_{32} - B_1T_{33},$$

$$4Hw_0w_2 = -B_2T_{11} + A_1T_{13} - B_1T_{21} + B_3T_{23} - A_3T_{31} + B_2T_{33},$$

$$4Hw_0w_3 = B_3T_{11} - A_1T_{12} + A_2T_{21} - B_3T_{22} + B_1T_{31} - B_2T_{32},$$

$$\begin{aligned}
4Hw_1^2 &= a_1T_{11} + 2b_3T_{12} + 2b_2T_{13} - a_1T_{22} - a_1T_{33} + a_1N , \\
4Hw_1w_2 &= a_2T_{12} + b_1T_{13} + a_1T_{21} + b_2T_{23} - b_3T_{33} + b_3N , \\
4Hw_1w_3 &= b_1T_{12} + a_3T_{13} - b_2T_{22} + a_1T_{31} + b_3T_{32} + b_2N , \\
4Hw_2^2 &= -a_2T_{11} + 2b_3T_{21} + a_2T_{22} + 2b_1T_{23} - a_2T_{33} + a_2N , \\
4Hw_2w_3 &= -b_1T_{11} + b_2T_{21} + a_3T_{23} + b_3T_{31} + a_2T_{32} + b_1N , \\
4Hw_3^2 &= -a_3T_{11} - a_3T_{22} + 2b_2T_{31} + 2b_1T_{32} + a_3T_{33} + a_3N .
\end{aligned}$$

The identities in the lemma are verified by direct (if rather lengthy) calculations.¹⁷ For the duration of verification we assume that the entries a_i and b_i of the matrix M and the coordinates w_k of w are formal variables; then the identities live in the polynomial ring $Z := \mathbb{Z}[(a_i), (b_i), (w_k)]$. This is proof by “passing through the generic portal”: the lemma is true for every integral domain R if and only if it is true for $R = Z$ (with very special choices of $M \in \mathbb{M}_3(Z)$ and $w \in Q_Z(M)$). It suffices to verify only four identities—say the first, the second, the fifth, and the sixth identity—since the other six identities follow by symmetry (just rotate the indices 1, 2, 3).

A **GCD domain** is an integral domain in which every pair of elements has a gcd (which is unique up to a unit multiplier). Let R be a GCD domain. Then

$$\gcd(ax_1, \dots, ax_n) = a \cdot \gcd(x_1, \dots, x_n)$$

(equality is up to a unit factor) for all $a, x_1, \dots, x_n \in R$.¹⁸ An element $x = (x_1, \dots, x_n)$ of the free R -module R^n is said to be **primitive** if $x = ry$ with $r \in R$ and $y \in R^n$ implies $r \in R^\times$; x is primitive iff $\gcd(x_1, \dots, x_n) = 1$.

Proposition 27. *Let R be a GCD domain and K the field of fractions of R . Let M be a symmetric nonsingular 3×3 matrix with entries in R , and w a primitive element of $Q := Q_R(M)$ with $N(w) \neq 0$. If the automorphism $x \mapsto wxw^{-1}$ of the K -algebra Q_K induces an automorphism of the R -algebra Q , then $N(w)$ divides $4 \det(M)$.*

Proof. The identities in Lemma 26 tell us that $N(w)$ divides

$$\gcd\{4 \det(M)w_iw_j \mid 0 \leq i, j \leq 3\} = 4 \det(M) \cdot \gcd\{w_iw_j \mid 0 \leq i, j \leq 3\} ,$$

where the gcd on the right hand side equals $\gcd(w_0, w_1, w_2, w_3)^2 = 1$. □

¹⁷Don't do it by hand: just feed the (alleged) identities to MATHEMATICA.

¹⁸Much more is true. If K is a field of fractions of a GCD domain R , then the group $G := K^\times/R^\times$, ordered by divisibility over R (for $x, y \in G$, $x \leq y$ iff $rx = y$ for some $r \in R^\bullet$), is a lattice-ordered group. See Chapter 2 in Stuart A. Steinberg, *Lattice-ordered Rings and Modules*, Springer, New York, 2010.

Let R be an integral domain, K its field of fractions, and $M \in \mathbb{M}_3(R)$ nonsingular symmetric. The matrix M determines a “classic” quadratic functional $q = q_M: R^3 \rightarrow R$, where $q(x) = x^\top M x$ for $x \in R^3$. We associate with the quadratic R -module (R^3, q) the quaternion R -algebra $Q := Q_R(M)$. We do this because the special orthogonal group of q is isomorphic to the special orthogonal group of ν_Q . In order to see this we identify R -linear transformations of R^3 with their matrices relative to the standard basis; then $\mathrm{SO}(q)$ is the set of all matrices $A \in \mathrm{GL}_3(R)$ with $\det A = 1$ such that $A^\top M A = M$, while $\mathrm{SO}(\nu_Q)$ is the set of all matrices $B \in \mathrm{GL}_3(R)$ with $\det B = 1$ such that $B^\top \tilde{M} B = \tilde{M}$. Inverting $A^\top M A = M$ (in $\mathbb{M}_3(K)$) and multiplying by $\det M$ we get $(A^{-\top})^\top \tilde{M} A^{-\top} = \tilde{M}$; ¹⁹ in the other direction we divide $B^\top \tilde{M} B = \tilde{M}$ by $\det M$ then invert, and obtain $(B^{-\top})^\top M B^{-\top} = M$. We therefore have the isomorphism of groups $\mathrm{SO}(q) \rightarrow \mathrm{SO}(\nu_Q) : A \mapsto A^{-\top}$ with the inverse $\mathrm{SO}(\nu_Q) \rightarrow \mathrm{SO}(q) : B \mapsto B^{-\top}$.

The most useful feature of $\mathrm{SO}(\nu_Q)$ is, of course, that it is isomorphic to $\mathrm{Aut}(Q)$. Let $B \in \mathrm{SO}(\nu_Q)$. There exists $w \in Q_R(M)$ with $N(w) \neq 0$, such that $B = T_w/N(w)$, where T_w is the matrix of the R -linear transformation $x \mapsto w x w^*$ of \vec{Q} (and $N(w)$ divides every entry of T_w). Then $B^{-\top} = T_{w^*}^\top/N(w) \in \mathrm{SO}(q)$ (where $N(w)$ divides every entry of the matrix $T_{w^*}^\top$). Therefore, having a general form of matrices in $\mathrm{SO}(\nu_Q)$, we have also a general form of matrices in $\mathrm{SO}(q)$.

We can exploit the chain of isomorphisms $\mathrm{SO}(q) \cong \mathrm{SO}(\nu_Q) \cong \mathrm{Aut}(Q)$ when we study the special orthogonal group $\mathrm{SO}(q)$ — its structure, its action on R^3 , etc. Since there are only so many interesting results that can be obtained for a general integral domain R , the enquiry soon has to be specialized to, say, a principal ideal domain R , or even further to $R = \mathbb{Z}$ (the ultimately classical special case), or to $R = F[X]$ with F a field of characteristic not 2 and X a formal variable.

And then the serious fun begins...

But that is another story.

16 Not an epilogue

We ran and ran after the stone that rolled and bounced down the slope. It dislodged other stones, which in turn dislodged more stones... so now we are chasing a small avalanche of stones. The bottom of the valley, obscured by roiling mists, is still very far below. How and where and when will it all end? Will it ever?

¹⁹Here $A^{-\top} = (A^{-1})^\top = (A^\top)^{-1}$. If $A \in \mathrm{GL}_3(R)$, then $A^{-\top} \in \mathrm{GL}_3(R)$; the mapping $A \mapsto A^{-\top}$ is an involution and an automorphism of $\mathrm{GL}_3(R)$.